# Polarization of Light
# (Part - I)

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## Contents



## References :

1. *Optics* by Ajoy Ghatak
2. *Introduction to Electrodynamics* by D. J. Griffiths
3. *Optics* by E. Hecht

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## 1 Introduction

► Light is electromagnetic (EM) wave which is transverse in nature. Electric field $\boldsymbol{E}(\boldsymbol{r}, t)$ and magnetic field $\boldsymbol{B}(\boldsymbol{r}, t)$ in EM wave oscillate in mutually perpendicular directions; furthermore,

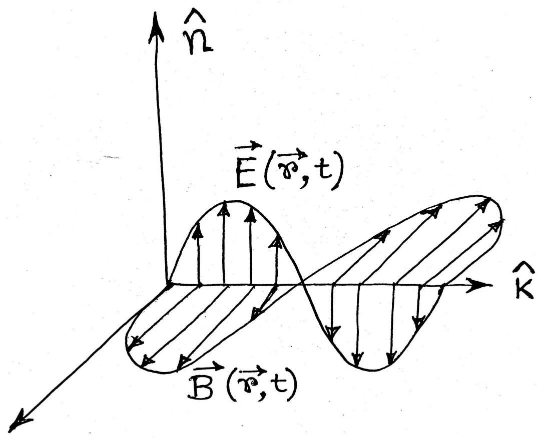


Figure 1: *A plane progressive EM wave.* $\hat{\boldsymbol{n}}$ *is the direction of electric field vector and* $\hat{\boldsymbol{k}}$ *is the propagation direction of EM wave.*

the wave propagation direction (i.e, the direction of the wave vector $\boldsymbol{k}$) is also perpendicular to $\boldsymbol{E}$ and $\boldsymbol{B}$ fields. The fields in a plane progressive EM wave can be represented by,

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{r},t) &= \boldsymbol{E}_0 \exp[i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)] \\ &= \hat{\boldsymbol{n}} E_0 \exp[i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)] \\ &\text{and} \\ \boldsymbol{B}(\boldsymbol{r},t) &= \boldsymbol{B}_0 \exp[i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)] \end{aligned} \tag{1}$$

Here, the **wave propagation velocity** is given by,

$$v_w = \frac{\omega}{|\boldsymbol{k}|} = \frac{\omega}{k}$$

The spatial variation of EM fields in a plane progressive EM wave is depicted in figure - 1; in which it is clear the $\boldsymbol{E}$, $\boldsymbol{B}$ and $\boldsymbol{k}$ are mutually perpendicular to each other.

In EM wave, $\boldsymbol{E}$ and $\boldsymbol{B}$ fields are interrelated to each other by the relation,

$$\boldsymbol{B}(\boldsymbol{r},t) = \frac{1}{c}\,[\hat{\boldsymbol{k}} \times \boldsymbol{E}(\boldsymbol{r},t)]$$

The strength of magnetic field is $c$ times smaller than the electric field strength. As the speed of light $c$ is very large number, the magnetic field strength is quite small compared to electric field strength.

► The **direction of polarization** of EM wave or light is considered as the direction of electric field vector. Any source of light contains very large number of atomic sources. Due to electron transition from higher energy levels to lower energy levels in the atoms, EM wave packets are emitted from these atoms. The emission process is random and there is no correlation among the atoms. Therefore, the wave pulses coming from individual atoms have no correlation and as a result the emitted EM waves may have random polarization directions with respect to each other. That means, overall light emitted form normal source is **unpolarized wave**, in which polarization direction randomly distributed. However, for each wave packet emitted form atomic emitter must obey the the condition : $\boldsymbol{E} \perp \boldsymbol{B} \perp \boldsymbol{k}$. If in EM wave the polarization direction does not show any random orientation then the wave is said to be **polarized wave**, later we can see that polarized wave can be classified in few types.

► While EM wave propagates through any material medium, it's electric field interacts with the atoms, molecules of the material; more specifically interacts with the charge distribution of the material. The electric field of EM wave induces macroscopic dipole moment inside the material. For linear material, the induced dipole moment is linearly proportional to the strength of electric field. In linear medium, the induced dipole moment per unit volume related to the electric field as,

$$\boldsymbol{P}(\boldsymbol{r}, t) = \epsilon_0 \chi_e(\omega) \boldsymbol{E}(\boldsymbol{r}, t)$$

where, the **susceptibility** $\chi_e(\omega)$ is in general function of frequency $\omega$. The frequency dependent **permittivity** $\epsilon(\omega)$ can be represented by,

$$\epsilon(\omega) = \epsilon_0(1 + \chi_e(\omega))$$

From the basic theory of electrodynamics, in particular from the Maxwell's equations, it can be shown that the speed of EM wave in material is,

$$v = \frac{1}{\sqrt{\mu_0 \epsilon(\omega)}} = \frac{1}{\sqrt{\mu_0 \epsilon_0 (1 + \chi(\omega))}} = \frac{c}{\sqrt{1 + \chi(\omega)}}$$

Hence the refractive index becomes,

$$n(\omega) = \frac{c}{v} = \sqrt{1 + \chi(\omega)}$$

$\chi_e$ physically represents the response of the material on application of electric field. It depends on the charge distribution in the material and depending on the charge distribution, $\chi_e$ can exhibit anisotropic variation. That means, in different directions in the material its values are different. The system (generally crystalline material) in which such anisotropy is observed is called as **anisotropic material**. For example **calcite** crystal is an anisotropic crystal. As the speed of light depends on $\chi_e$, in anisotropic material speed of light is different along different directions.

# 2 Anisotropic material

In this medium, the electronic properties, in particular, the **permittivity** or the **susceptibility** exhibits anisotropic variations. **Uniaxial** crystal is a kind of anisotropic medium. In these kind of crystals, the permittivity $\epsilon$ is different along a particular direction (fixed to the crystal) and any perpendicular direction to this reference direction, $\epsilon$ remain same. The direction along which $\epsilon$ is different is called **optic axis** of the crystal. Later we will see that optic axis can be defined in an alternative way in terms of speed of light of two different polarization components. Since, the system has only one such direction, it is called uniaxial crystal. There exist **biaxial crystal** also; for which we can identify two optic axis. Mathematical theory of biaxial crystal is quite complicated. Following we consider the propagation of EM wave through uniaxial crystal to understand how polarization of light explicitly related to the anisotropy of crystal.

## 2.1 Propagation of light in uniaxial crystal

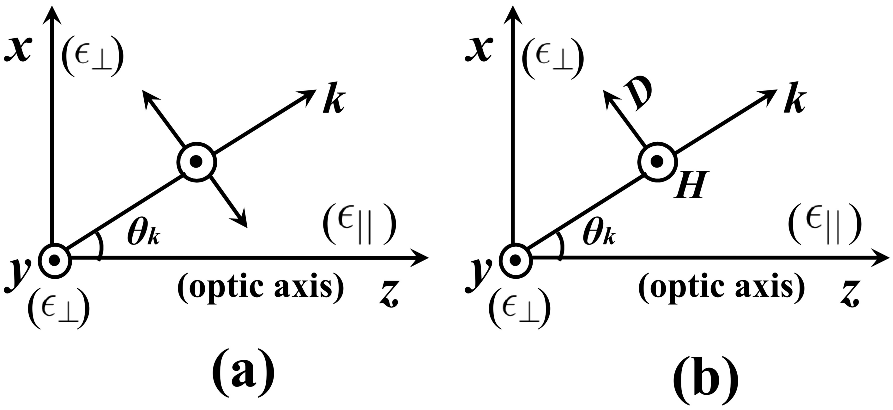


Figure 2: (In both) Let set the $z-$axis parallel to optic axis. $\epsilon_{||}$ is the permittivity along optic axis and the same perpendicular to optic axis is $\epsilon_\perp$. Wave is propagating with wave vector $\boldsymbol{k}$ making an angle $\theta_k$ with optic axis. (a) $\boldsymbol{D}$ field is decomposed in two components : $y$-component and $x-z$ component. (b) $x-z$ component of $\boldsymbol{D}$ field is considered with $\boldsymbol{H}$ field normal to both $\boldsymbol{D}$ and $\boldsymbol{k}$.

For simplicity we consider our Cartesian $z$-axis to coincide with the optic axis of the crystal. The anisotropy of pemittivity is given by,

$$\epsilon_x = \epsilon_y = \epsilon_\perp \neq \epsilon_z = \epsilon_{||}$$

where, we consider $\epsilon_{||}$ to be the permittivity along the optic axis and $\epsilon_\perp$ to be the permittivity along any perpendicular direction of optic axis. According to our coordinate system, the system is isotropic on any $x-y$ plane.

Furthermore, we assume that the system is non-magnetic, therefore, it's permeability is taken to be $\mu_0$. The Maxwell's equations in the material (no free charge and free current) can be expressed as,

$$\begin{aligned}
&\text{i)} \quad \boldsymbol{\nabla} \cdot \boldsymbol{D}(\boldsymbol{r}, t) = 0 \\
&\text{ii)} \quad \boldsymbol{\nabla} \cdot \boldsymbol{B}(\boldsymbol{r}, t) = 0 \\
&\text{iii)} \quad \boldsymbol{\nabla} \times \boldsymbol{E}(\boldsymbol{r}, t) = -\frac{\partial}{\partial t}\boldsymbol{B}(\boldsymbol{r}, t) \\
&\text{iv)} \quad \boldsymbol{\nabla} \times \boldsymbol{H}(\boldsymbol{r}, t) = \frac{\partial}{\partial t}\boldsymbol{D}(\boldsymbol{r}, t)
\end{aligned}$$

From these Maxwell's equations one can derive the wave equation for EM fields. The space-time variation of fields in a plane progressive EM wave can be represented by (1); and similar expression can be wrote down for $\boldsymbol{D}$ and $\boldsymbol{H}$.

In case of isotropic crystal $\epsilon$ is independent of direction in the crystal. In such cases,

$$\boldsymbol{D}(\boldsymbol{r}, t) = \epsilon \boldsymbol{E}(\boldsymbol{r}, t)$$

that means $\boldsymbol{D}$ and $\boldsymbol{E}$ are parallel to each other. However, for anisotropic medium, let the unaxial crystal depicted in figure - 2,

$$D_x = \epsilon_x E_x = \epsilon_\perp E_x \quad ; \quad D_y = \epsilon_y E_y = \epsilon_\perp E_y \quad ; \quad D_z = \epsilon_z E_z = \epsilon_{||} E_z$$

In more compact form,

$$\begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} = \begin{pmatrix} \epsilon_\perp & 0 & 0 \\ 0 & \epsilon_\perp & 0 \\ 0 & 0 & \epsilon_{||} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

**That means, in anisotropic crystal, $\boldsymbol{D}$ and $\boldsymbol{E}$ fields are not parallel to each other.**

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**Exercise 1.** Show that in anisotropic crystal, $\boldsymbol{D}$ field is perpendicular to the wave vector $\boldsymbol{k}$.
**Solution :**
For plane progressive wave,

$$\boldsymbol{D}(\boldsymbol{r}, t) = \boldsymbol{D}_0 \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)]$$

where;

$$\boldsymbol{k} = \hat{\boldsymbol{x}} k_x + \hat{\boldsymbol{y}} k_y + \hat{\boldsymbol{z}} k_z \quad \text{and} \quad \boldsymbol{r} = \hat{\boldsymbol{x}} x + \hat{\boldsymbol{y}} y + \hat{\boldsymbol{z}} z$$

Consider the Maxwell's equation,

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = 0$$

Now consider the operation,

$$\begin{aligned} \frac{\partial}{\partial x} \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)] &= \frac{\partial}{\partial x} \exp[i(x k_x + y k_y + z k_z - \omega t)] \\ &= i k_x \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)] \end{aligned}$$

Similarly,

$$\frac{\partial}{\partial y} \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)] = i k_y \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)]$$

and

$$\frac{\partial}{\partial z} \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)] = i k_z \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)]$$

Now,

$$\begin{aligned} & \boldsymbol{\nabla} \cdot \boldsymbol{D} = 0 \\ \Rightarrow \quad & \left( \hat{\boldsymbol{x}} \frac{\partial}{\partial x} + \hat{\boldsymbol{y}} \frac{\partial}{\partial y} + \hat{\boldsymbol{z}} \frac{\partial}{\partial z} \right) \cdot \underbrace{(\hat{\boldsymbol{x}} D_{0x} + \hat{\boldsymbol{y}} D_{0y} + \hat{\boldsymbol{z}} D_{0z})}_{\boldsymbol{D}_0} \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)] = 0 \\ \Rightarrow \quad & i(k_x D_{0x} + k_y D_{0y} + k_z D_{0z}) \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)] = 0 \\ \Rightarrow \quad & i \boldsymbol{k} \cdot \boldsymbol{D} = 0 \end{aligned} \tag{2}$$

**That means $\boldsymbol{D}$ field is perpendicular to wave vector $\boldsymbol{k}$.**

According to the figure - 2(a), we can consider two components of polarization for an EM wave propagating along $\boldsymbol{k}$.

i) Polarization normal to the plane formed by $\boldsymbol{k}$ and optic axis (hence $x - z$ plane), we call this polarization component as out-of-plane polarization or the $y$-polarization (according to our choice of coordinate system).

ii) Another component of polarization is considered to be lying in the plane of $\boldsymbol{k}$ and optic axis (hence $x - z$ plane). We call this polarization component as in-plane polarization.

• To be noted that, for out-of-plane polarized wave, $\boldsymbol{D}$ is always pointing along perpendicular direction of the optic axis irrespective of propagation direction $\hat{\boldsymbol{k}}$. So, this particular polarization component always experiences $\epsilon_\perp$ irrespective of its propagation direction $\hat{\boldsymbol{k}}$. Therefore, the speed of propagation of this wave is independent of its propagation direction. We call this light wave as **ordinary wave** because it behaves like ordinary light wave propagation through isotropic medium. Let,

$$v_0 = \text{ ordinary wave velocity}$$

Then,

$$\boxed{v_0 = \frac{1}{\sqrt{\mu_0 \epsilon_\perp}}}$$

• In case of in-plane polarization component, $\boldsymbol{D}$ experiences both $\epsilon_\perp$ and $\epsilon_{||}$ (unless the propagation direction is either along $x$-axis or $z$-axis). Depending on the direction of propagation (angle $\theta_k$), effect of $\epsilon_\perp$ and $\epsilon_{||}$ can vary which in effect gives rise to variation speed of EM wave along different propagation direction $\theta_k$. Which is unusual in general cases, so, we call this light wave as **extra ordinary wave**. The in-plane polarization component would exhibit some interesting phenomena as it experiences the anisotropy of permittivity $\epsilon$. Let's focus on the in-plane polarization.

### 2.1.1 Extraordinary wave velocity

• Consider in figure - 2(b) the in-plane polarized light is traveling along $x-$axis (i.e, $\theta_k = \pi/2$). Then, its polarization is along $z-$axis, that means the polarization experiences $\epsilon_{||}$. Hence the light propagation velocity,

$$v_e = \frac{1}{\sqrt{\mu_0 \epsilon_{||}}}$$

where,

$$v_e = \text{ extra ordinay wave velocity}$$

• If the in-plane polarized light propagates along optic axis (i.e, $\theta_k = 0$), then it is polarized along $x-$axis and experiences $\epsilon_\perp$. The wave propagation speed will be,

$$\frac{1}{\sqrt{\mu_0 \epsilon_\perp}} = v_0 \quad ; \text{ idetified as ordinary wave velocity}$$

- **Extra ordinary light wave is polarized in the plane formed by $\boldsymbol{k}$ and optic axis. Ordinary light wave is polarized in the normal direction of the plane formed by $\boldsymbol{k}$ and optic axis**
- **Extra ordinary light and ordinary light both travel with same speed of $\boldsymbol{v}_o$ along the optic axis.**

- According to our diagram - 2(b), for in-plane polarized wave, $\boldsymbol{D} = (D_x, D_z)$ and $\boldsymbol{E} = (E_x, E_z)$. Their relation,

$$\begin{pmatrix} D_x \\ D_z \end{pmatrix} = \begin{pmatrix} \epsilon_\perp & 0 \\ 0 & \epsilon_{||} \end{pmatrix} \begin{pmatrix} E_x \\ E_z \end{pmatrix} \tag{3}$$

For plane progressive EM wave,

$$\boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{E}_0 \exp[i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)] \qquad \text{and} \qquad \boldsymbol{B}(\boldsymbol{r}, t) = \boldsymbol{B}_0 \exp[i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]$$

$$\boldsymbol{D}(\boldsymbol{r}, t) = \boldsymbol{D}_0 \exp[i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)] \qquad \text{and} \qquad \boldsymbol{H}(\boldsymbol{r}, t) = \boldsymbol{H}_0 \exp[i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]$$

Now, consider the Maxwell's equation

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

Now from exercise - 2;

$$\Rightarrow \quad i(\boldsymbol{k} \times \boldsymbol{E}) = i\omega \boldsymbol{B} = i\omega\mu_0 \boldsymbol{H} \quad \text{; non-magnetic material}$$

$$\Rightarrow \quad \boldsymbol{H} = \frac{1}{\omega\mu_0}(\boldsymbol{k} \times \boldsymbol{E})$$

Consider the another Maxwell's equation,

$$\begin{aligned}
&\boldsymbol{\nabla} \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t} \\
\Rightarrow \quad & i(\boldsymbol{k} \times \boldsymbol{H}) = -i\omega \boldsymbol{D} \\
\Rightarrow \quad & \boldsymbol{D} = -\frac{1}{\omega}(\boldsymbol{k} \times \boldsymbol{H}) \\
\Rightarrow \quad & \boldsymbol{D} = -\frac{1}{\omega}\boldsymbol{k} \times \underbrace{\left(\frac{1}{\omega\mu_0}\boldsymbol{k} \times \boldsymbol{E}\right)}_{\boldsymbol{H}} \\
\Rightarrow \quad & \boldsymbol{D} = -\frac{1}{\mu_0\omega^2}[\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{E})] \\
\Rightarrow \quad & \boldsymbol{D} = \frac{1}{\mu_0\omega^2}[k^2\boldsymbol{E} - \boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{E})]
\end{aligned}$$

For in-plane polarized wave, $\boldsymbol{k} = (k_x, k_z)$. The components are; $k_x = |\boldsymbol{k}|\sin\theta_k = k\sin\theta_k$ and $k_z = |\boldsymbol{k}|\cos\theta_k = k\cos\theta_k$ (according to figure - 2(b)).
We have obtained above,

$$\boldsymbol{D} = \frac{1}{\mu_0\omega^2}[k^2\boldsymbol{E} - \boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{E})]$$

Now, the $x-$component of in-plane polarized wave,

$$\begin{aligned}
& D_x = \frac{1}{\mu_0\omega^2}[k^2E_x - k_x(k_xE_x + k_zE_z)] \\
\Rightarrow \quad & \underbrace{\epsilon_\perp E_x}_{D_x} = \frac{k^2}{\mu_0\omega^2}(E_x - E_x\sin^2\theta_k - E_z\sin\theta_k\cos\theta_k) \\
& \text{Now : } v_0 = 1/\sqrt{\mu_0\epsilon_\perp} \\
& \text{Wave velocity along } \boldsymbol{k} : \ v_w(\theta_k) = \omega/k \\
\Rightarrow \quad & (v_w^2(\theta_k) - v_0^2\cos^2\theta_k)E_x + (v_0^2\cos\theta_k\sin\theta_k)E_z = 0 \qquad (4)
\end{aligned}$$

Similarly for $z$-component of in-plane polarized light,

$$\begin{aligned}
& D_z = \frac{1}{\mu_0\omega^2}[k^2E_z - k_z(k_xE_x + k_zE_z)] \\
\Rightarrow \quad & \underbrace{\epsilon_{||} E_z}_{D_z} = \frac{k^2}{\mu_0\omega^2}[E_z - E_z\cos^2\theta_k - E_x\cos\theta_k\sin\theta_k] \\
& \text{Now : } v_e = 1/\sqrt{\mu_0\epsilon_{||}} \\
\Rightarrow \quad & (v_w^2(\theta_k) - v_e^2\sin^2\theta_k)E_z + (v_e^2\cos\theta_k\sin\theta_k)E_x = 0 \qquad (5)
\end{aligned}$$

For non-trivial values of $E_x$ and $E_z$ the determinate of the coefficients of $E_x$ and $E_z$ of equations (4) and (5) must vanish.
Therefore,

$$\begin{vmatrix} v_w^2 - v_0^2\cos^2\theta & v_0^2\cos\theta_k\sin\theta_k \\ v_e^2\cos\theta_k\sin\theta_k & v_w^2 - v_e^2\sin^2\theta_k \end{vmatrix} = 0$$

From here it is easy to find out the relation;

$$\boxed{v_w^2(\theta_k) = v_0^2\cos^2\theta_k + v_e^2\sin^2\theta_k} \qquad (6)$$

where,
$v_w(\theta_k)$ = Extra ordinary wave velocity in the direction $\theta_k$ with respect to optic axis (figure - 2)
$v_0 = 1/\sqrt{\mu_0\epsilon_\perp}$ = Ordinary wave velocity along optic axis
$v_e = 1/\sqrt{\mu_0\epsilon_{||}}$ = Extra ordinary wave velocity perpendicular to optic axis.

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**Exercise 2.** For $\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E}_0\exp[i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]$, show that $\boldsymbol{\nabla}\times\boldsymbol{E} = i(\boldsymbol{k}\times\boldsymbol{E})$
**Hints :**
Find the partial derivative like,

$$\frac{\partial}{\partial x}\exp[i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]$$

for each $x, y, z$ components (see exercise - 1).

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## 2.2 Extraordinary wave and extra ordinary ray

Consider the figure - 2(b) in which it is represented that extra ordinary light wave[1] is propagating with wave vector $\boldsymbol{k}$ making an angle $\theta_k$ with the optic axis. For this wave, $\boldsymbol{H}$ field is pointing

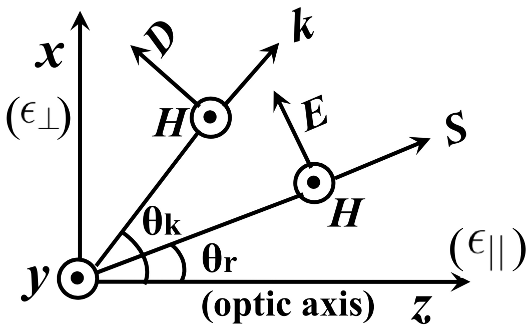


Figure 3: **Figure for extraordinary light :** *Wave propagation direction $\theta_k$ ( direction of wave vector $\boldsymbol{k}$) and ray propagation direction $\theta_r$ (direction of $\boldsymbol{S}$) are not same in anisotropic medium. $\boldsymbol{E}$ is not parallel to $\boldsymbol{D}$. Here, $\boldsymbol{D} \perp \boldsymbol{H} \perp \boldsymbol{k}$ and $\boldsymbol{E} \perp \boldsymbol{H} \perp \boldsymbol{S}$.*

out of the $x - z$ plane (i.e, plane formed by $\boldsymbol{k}$ and optic axis). Hence, three vectors are mutually perpendicular to each other : $\boldsymbol{D} \perp \boldsymbol{k} \perp \boldsymbol{H}$.
If,
$\hat{\boldsymbol{k}}$ = unit vector along wave propagation direction
$\hat{\boldsymbol{d}}$ = unit vector along $\boldsymbol{D}$
$\hat{\boldsymbol{h}}$ = unit vector along $\boldsymbol{H}$
Then, from the geometry,

$$\hat{\boldsymbol{k}} = \hat{\boldsymbol{d}} \times \hat{\boldsymbol{h}}$$

**Light ray propagation direction is the direction along which EM wave energy flows, i.e, the direction of Poynting vector** which is defined as,

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$$

or $\quad \hat{\boldsymbol{s}} = \hat{\boldsymbol{e}} \times \hat{\boldsymbol{h}}$
where, the later expressions are of the respective unit vectors.
In anisotropic crystal, it is already noted that $\boldsymbol{D}$ is not parallel to $\boldsymbol{E}$. So, **in anisotropic crystal wave propagation direction $\hat{\boldsymbol{k}}$ $(= \hat{\boldsymbol{d}} \times \hat{\boldsymbol{h}})$ and ray propagation direction $\hat{\boldsymbol{s}}$ $(= \hat{\boldsymbol{e}} \times \hat{\boldsymbol{h}})$ are not same**.

The wave propagation and ray propagation associated with extraordinary light in a uniaxial crystal is schematically depicted in figure - 3

### 2.2.1 Extarordinary wave and extarordinary ray propagation direction

Following we identify that ray propagation direction and wave propagation direction are related to each other.

[1]Already mentioned that extra ordinary light is polarized in the plane of $\boldsymbol{k}$ and optic axis

**The ray velocity ($v_r$) and wave velocity ($v_w$) are defined as,**

$$\boxed{v_r = \frac{|\boldsymbol{S}|}{u} \qquad \text{and} \qquad v_w = \frac{\omega}{|\boldsymbol{k}|}}$$

where,

$$\text{Poynting vector :} \quad \boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$$

$$\text{Energy density :} \quad u = \frac{1}{2}(\boldsymbol{D} \cdot \boldsymbol{E} + \boldsymbol{B} \cdot \boldsymbol{H})$$

Now,

$$\boldsymbol{H} = \boldsymbol{H}_0 \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)] \qquad \text{and} \qquad \boldsymbol{D} = \boldsymbol{D}_0 \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)]$$

$$\boldsymbol{E} = \boldsymbol{E}_0 \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)] \qquad \text{and} \qquad \boldsymbol{B} = \boldsymbol{B}_0 \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)]$$

Putting these expressions in the following Maxwell's equations,

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t} \qquad \text{and} \qquad \boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

We can obtain (with the help of exercise - 2),

$$\boldsymbol{D} = \frac{1}{\omega}(\boldsymbol{H} \times \boldsymbol{k}) \qquad \text{and} \qquad \boldsymbol{B} = \frac{1}{\omega}(\boldsymbol{k} \times \boldsymbol{E})$$

Now put the expressions of $\boldsymbol{D}$ and $\boldsymbol{B}$ in the energy density expression,

$$\begin{aligned} u &= \frac{1}{2}(\boldsymbol{D} \cdot \boldsymbol{E} + \boldsymbol{B} \cdot \boldsymbol{H}) \\ &= \frac{1}{2\omega}[\boldsymbol{E} \cdot (\boldsymbol{H} \times \boldsymbol{k}) + \boldsymbol{H} \cdot (\boldsymbol{k} \times \boldsymbol{E})] \\ &= \frac{1}{\omega}\boldsymbol{k} \cdot (\boldsymbol{E} \times \boldsymbol{H}) \\ &= \frac{\boldsymbol{k} \cdot \boldsymbol{S}}{\omega} \\ u &= \frac{1}{\omega}|\boldsymbol{k}||\boldsymbol{S}| \cos(\theta_k - \theta_r) \end{aligned}$$

Ray velocity along $\theta_r$,

$$v_r(\theta_r) = \frac{|\boldsymbol{S}|}{u} = \frac{\omega}{|\boldsymbol{k}| \cos(\theta_k - \theta_r)} \qquad ; \text{ using the above expression of } u$$

But, wave velocity, $v_w(\theta_k) = \omega/|\boldsymbol{k}|$

So, we have,

$$\boxed{v_r(\theta_r) = \frac{v_w(\theta_k)}{\cos(\theta_k - \theta_r)} \qquad \textbf{or} \qquad v_w(\theta_k) = v_r(\theta_r)\cos(\theta_k - \theta_r)} \tag{7}$$

Now according to figure - 3,

$$E_x = E \cos\theta_r \qquad \text{and} \qquad E_z = -E \sin\theta_r$$

$$D_x = D \cos\theta_k \qquad \text{and} \qquad D_z = -D \sin\theta_k$$

Also, for anisotropic medium with optic axis parallel to $z-$axis,

$$D_x = \epsilon_\perp E_x = \frac{E_x}{\mu_0 v_o^2} \qquad \text{and} \qquad D_z = \epsilon_{||} E_z = \frac{E_z}{\mu_0 v_e^2}$$

Since, as stated earlier,

$$v_o = \frac{1}{\sqrt{\mu_0 \epsilon_\perp}} \qquad \text{and} \qquad v_e = \frac{1}{\sqrt{\mu_0 \epsilon_{||}}}$$

So, we have,

$$\frac{D_z}{D_x} = \frac{v_o^2}{v_e^2}\frac{E_z}{E_x}$$
$$\Rightarrow \quad \tan\theta_k = \frac{v_o^2}{v_e^2}\tan\theta_r$$

Therefore,**with respect to optic axis, the relation between wave propagation direction $\boldsymbol{\theta_k}$ and ray propagation direction $\boldsymbol{\theta_r}$**,

$$\boxed{v_e^2 \tan\theta_k = v_o^2 \tan\theta_r} \tag{8}$$

▶ **Note :**

- Suppose, $\theta_r = 0$, i.e; ray is propagating along optic axis. Hence, obviously $\theta_k = 0$ (from relation (8)). That means, **along optic axis, wave vector $\boldsymbol{k}$ and light ray coincide.**
- Similarly, if $\theta_r = \pi/2$ i.e; ray is propagating perpendicular to optic axis. Hence also from relation (8), $\theta_k = \pi/2$. That means, **perpendicular to optic axis, wave vector $\boldsymbol{k}$ and light ray coincide.**

### 2.2.2 Extraordinary ray velocity

In this section we will develop the **relation of extraordinary ray velocity with its propagation direction $\boldsymbol{\theta_r}$.**
Consider the relation (7),

$$v_w(\theta_k) = v_r(\theta_r)\cos(\theta_k - \theta_r) \qquad ; \text{ Refer to figure - 3}$$

Or,

$$\begin{aligned}
\frac{1}{v_r^2} &= \frac{\cos^2(\theta_k - \theta_r)}{v_w^2} \\
&= \frac{(\cos\theta_k\cos\theta_r + \sin\theta_k\sin\theta_r)^2}{v_0^2\cos^2\theta_k + v_e^2\sin^2\theta_k} && \text{; Using relation (6) for expression of } v_w \\
&= \frac{\cos^2\theta_r(1+\tan\theta_r\tan\theta_k)^2}{v_o^2\left(1+\frac{v_e^2}{v_o^2}\tan^2\theta_k\right)} \\
&= \frac{\cos^2\theta_r\left(1+\frac{v_o^2}{v_e^2}\tan^2\theta_r\right)^2}{v_o^2\left(1+\frac{v_o^2}{v_e^2}\tan^2\theta_r\right)} && \text{; Usign expression for } \tan\theta_k \text{ form (8)} \\
&= \frac{\cos^2\theta_r}{v_o^2}\left(1+\frac{v_o^2}{v_e^2}\tan^2\theta_r\right) \\
&= \frac{\cos^2\theta_r}{v_o^2} + \frac{\sin^2\theta_r}{v_e^2}
\end{aligned}$$

Therefore, **with respect to optic axis along the direction of $\theta_r$ angle, the extraordinary ray velocity $v_r(\theta_r)$ is given by,**

$$\boxed{\frac{1}{v_r^2(\theta_r)} = \frac{\cos^2\theta_r}{v_o^2} + \frac{\sin^2\theta_r}{v_e^2}} \tag{9}$$

- **Extraordinary wave :**
  The wave propagation velocity at $\theta_k$ angle w.r.t optic axis,

$$v_w^2(\theta_k) = v_o^2\cos^2\theta_k + v_e^2\sin^2\theta_k$$

  Associated refractive index for extraordinary wave,

$$\frac{1}{n_w^2(\theta_k)} = \frac{\cos^2\theta_k}{n_o^2} + \frac{\sin^2\theta_k}{n_e^2}$$

  where,

$$n_w(\theta_k) = \frac{c}{v_w(\theta_k)} \qquad ; \qquad n_o = \frac{c}{v_o} \qquad ; \qquad n_e = \frac{c}{v_e}$$

- **Extraordinary ray :**
  The ray propagation velocity at $\theta_r$ angle w.r.t optic axis,

$$\frac{1}{v_r^2(\theta_r)} = \frac{\cos^2\theta_r}{v_o^2} + \frac{\sin^2\theta_r}{v_e^2}$$

  Associated refractive index for extraordinary ray,

$$n_r^2(\theta_r) = n_o^2\cos^2\theta_r + n_e^2\sin^2\theta_r$$

  where,

$$n_r(\theta_r) = \frac{c}{v_r(\theta_r)} \qquad ; \qquad n_o = \frac{c}{v_o} \qquad ; \qquad n_e = \frac{c}{v_e}$$

### 2.2.3 Shape of the extraordinary wave front

Emanating from a point source, in time interval $t$, the extraordinary ray traverses.

$tv_r(\theta_r)$ distance along $\theta_r$ w.r.t optic axis $= \rho$ (say).

$tv_o$ distance along optic axis $= a$ (say)

$tv_e$ distance perpendicular optic axis $= b$ (say)

For extraordinary ray,

$$\frac{1}{v_r^2(\theta_r)} = \frac{\cos^2\theta_r}{v_o^2} + \frac{\sin^2\theta_r}{v_e^2}$$

Or,

$$\frac{1}{\rho^2} = \frac{\cos^2\theta_r}{a^2} + \frac{\sin^2\theta_r}{b^2}$$

**This represents parametric form of ellipse.**

Suppose optic axis is along Cartesian $z-$axis, then we can consider the transformation,

$$z = \rho\cos\theta_r \qquad \text{and} \qquad x = \rho\sin\theta_r$$

Hence, we have the standard form of ellipse in $x - z$ plane,

$$\frac{z^2}{a^2} + \frac{x^2}{b^2} = 1$$

**The extraordinary wave front is elliptical in shape in the plane of wave vector *k* and optic axis. Any one of axes (either major or minor) of the ellipse coincides with the optic axis of the uniaxial crystal.**

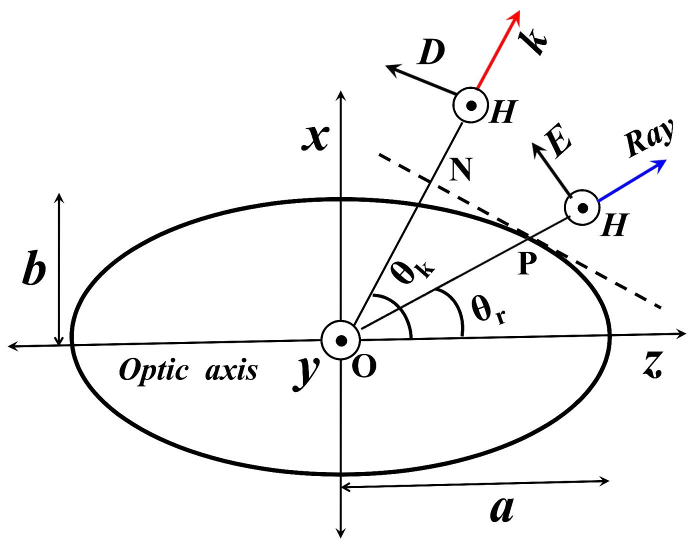


Figure 4:

► **The wave front is depicted in figure - 4. In this figure :**

- The ellipse represents the extraordinary wave front. Which is basically the trajectory of the tip of light ray at any arbitrary instant of time $t$.
- Wave vector $\boldsymbol{k}$ is normal to the tangent drawn (dotted line) on the ellipse at the intersection point by ray. Obviously, ray and wave vector are directed in same direction only along optic axis and perpendicular to optic axis.
- OP $= tv_r(\theta_r)$, ON $= tv_w(\theta_k)$, $a = tv_o$ and $b = tv_e$.
- From the diagram,

$$ON = OP\cos(\theta_k - \theta_r)$$

Or,

$$v_w = v_r\cos(\theta_k - \theta_r)$$

### 2.2.4 Positive crystal and negative crystal

Ordinary ray travels with velocity $v_o$ in every directions, but the extraordinary ray travels with velocity $v_o$ along optic axis and with velocity $v_e$ along perpendicular too optic axis. In some crystals, $v_o > v_e$ or $(n_o < n_e)$, these crystals are called **positive crystal**. On the other hand in some crystals, $v_o < v_e$ or $(n_o > n_e)$, these crystals are called **negative crystal**. The spherical wave front of ordinary light (**red**) and elliptical wave front of extraordinary light (**blue**) is depicted in figure - 5 for (a) positive crystal and (b) negative crystal.

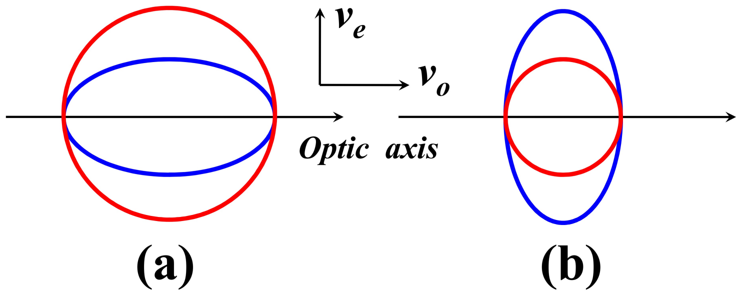


Figure 5: (a) Positive crystal $(v_o > v_e)$ and (b) negative crystal $(v_o < v_e)$. **Blue** elliptical wave front for extraordinary light and **Red** spherical wave front for ordinary light.

## 2.3 Huygens' construction of wave front in uniaxial crystal

Unpolarized light when enters in the uniaxial crystal, it splits into ordinary light and extraordinary light and these two light component propagate through the crystal. The propagation can be explained by constructing Huygens' wave front for these lights. To do this we should follow some basic properties of ordinary light and extraordinary light; which are already discussed earlier, still we summarize some of them here.

- **Some basic properties of ordinary light and extraordinary light :**
  - Ordinary light is polarized normal to the plane formed by wave vector and optic axis. Extraordinary light is polarized in the plane of wave vector and optic axis.
  - Ordinary light propagates with same speed $(v_o)$ along every directions. Extraordinary light ray velocity is direction dependent (see relation (9)).
  - Along optic axis, both ordinary ray and extraordinary ray velocity is same.
  - Ordinary light wave front is spherical in shape. Shape of extraordinary light wave front is ellipsoid of revolution.

► **Below some figures are drawn for various cases of EM wave propagation in uniaxial crystal. In all these figures - 6 to 9, blue colour indicates extraordinary wave front (elliptical shape) and red colour indicates ordinary wave front (circular shape).**

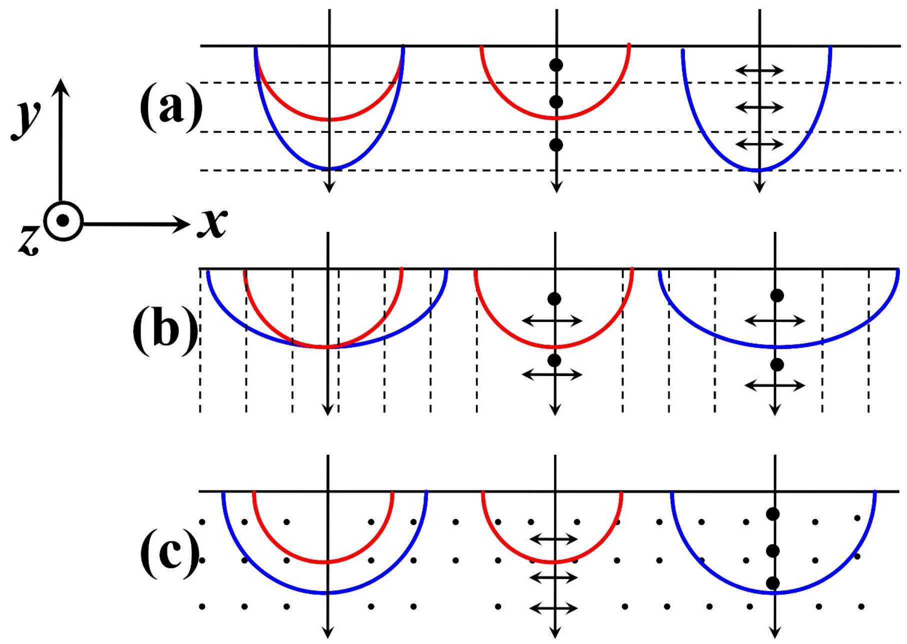


Figure 6: **Blue** colour indicates extraordinary wave front (elliptical shape) and **Red** colour indicates ordinary wave front (circular shape). **(a)** Optic axis (dotted lines) parallel to interface and in the plane of incidence. **(b)** Optic axis normal to the interface and in the plane of incidence. **(c)** Optic axis parallel to the interface and normal to the plane of incidence.

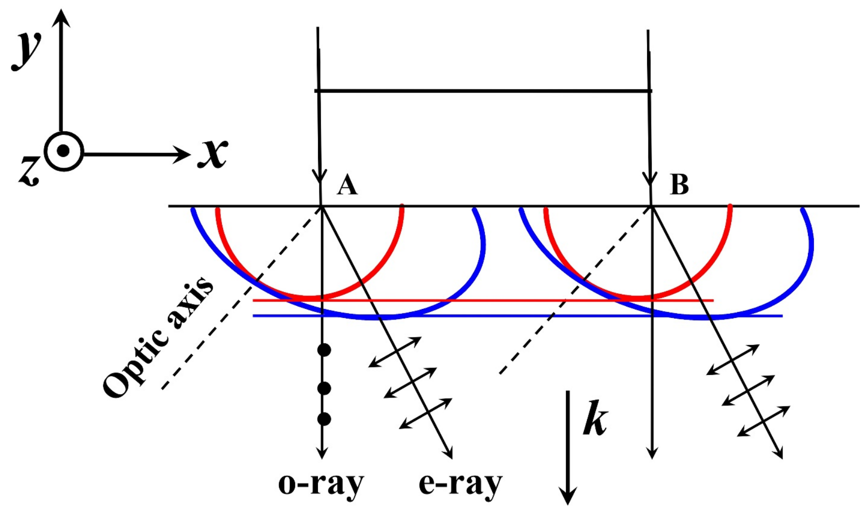


Figure 7: **Blue** colour indicates extraordinary wave front (elliptical shape) and **Red** colour indicates ordinary wave front (circular shape). Normal incidence of light. Optic axis inclined with the interface and in the plane of incidence ($x-y$ plane). Note, extraordinary light ray deviates though the incidence angle is zero.

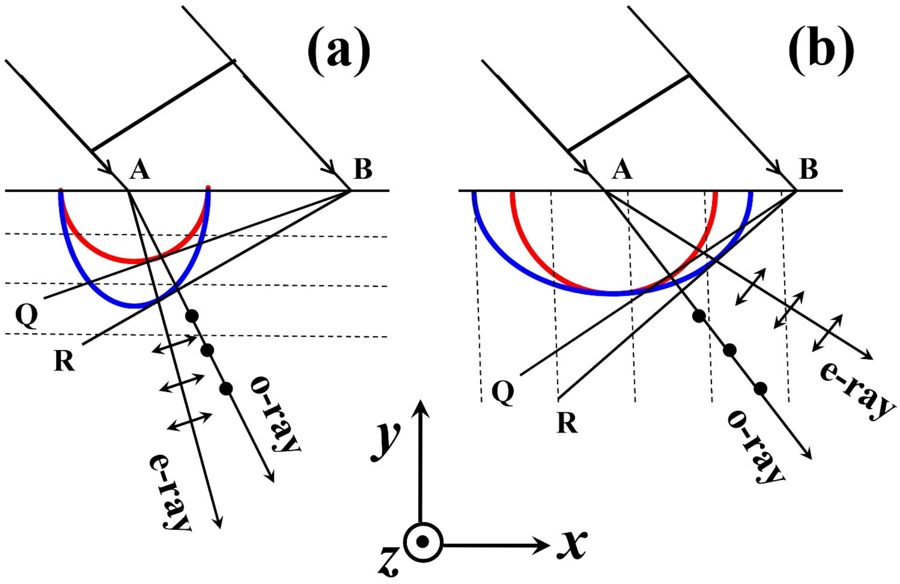


Figure 8: **Blue** colour indicates extraordinary wave front (elliptical shape) and **Red** colour indicates ordinary wave front (circular shape). Oblique incidence of light. **(a)** Optic axis parallel to interface and in the plane of incidence ($x-y$ plane). **(b)** Optic axis normal to interface and in the plane of incidence ($x-y$ plane). In both cases, BQ tangent to ordinary wave front, BR tangent to extraordinary wave front.

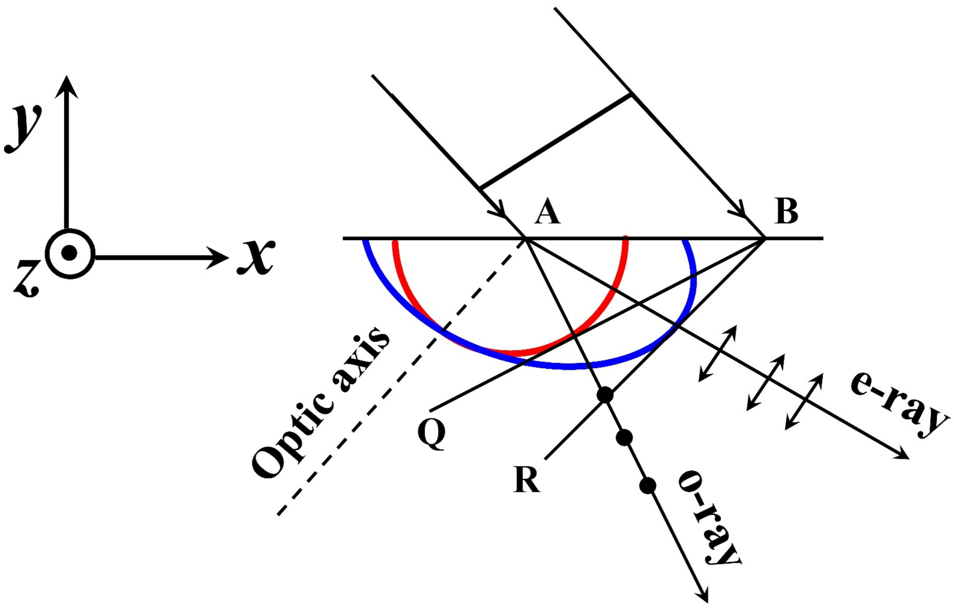


Figure 9: **Blue** colour indicates extraordinary wave front (elliptical shape) and **Red** colour indicates ordinary wave front (circular shape). Oblique incidence of light. Optic axis is inclined with the interface and in the plane of incidence ($x-y$ plane). BQ tangent to ordinary wave front, BR tangent to extraordinary wave front.

# Polarization of Light

## (Part - II)

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# Contents :

## 3. Classification of polarized light :

It is already discussed that light emitted from source remain unpolarized or in other words, randomly polarized. It is possible to convert such unpolarized light into polarized light.

Basically, polarized lights are classified as;

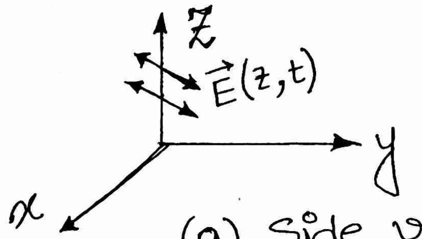
i) Linearly polarized or plane polarized
ii) Elliptically polarized
iii) Circularly polarized.

In case of linealy polarized light (or the plane polarized light), electric field vector in EM wave lies in a fixed plane. For elliptically polarized light, the electric field vector traverses an elliptical path and similar for circularly polarized light. Depending on the direction of rotation of electric field vector on elliptical path or circular path, the polarized light is assigned to be

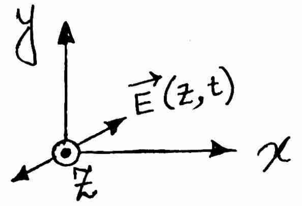
a) Left elliptically polarized (LEP)
b) Right " " (REP)
c) Left circularly polarized (LCP)
d) Right " " (RCP)

Below we represent the diagrams of various types of polarized lights.

Consider light wave is propagating along z-axis.

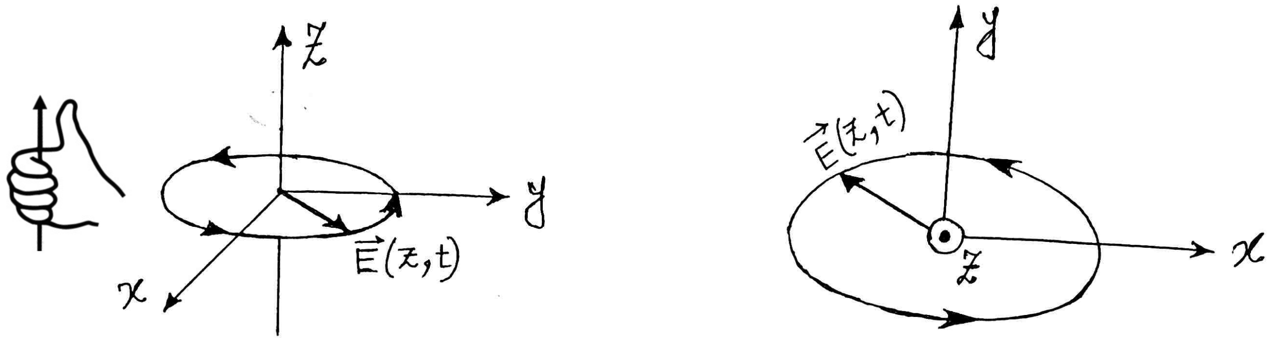


(a) Side view



(b) Top view.

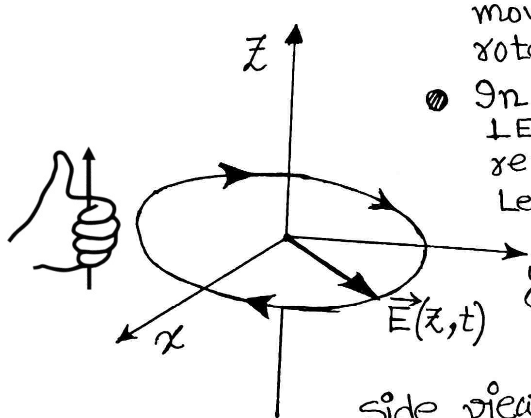
Figure – 17.1 : Linearly polarized wave
$\vec{E}(z,t)$ is the expression for $\vec{E}$ field, wave propagation direction is z-axis.



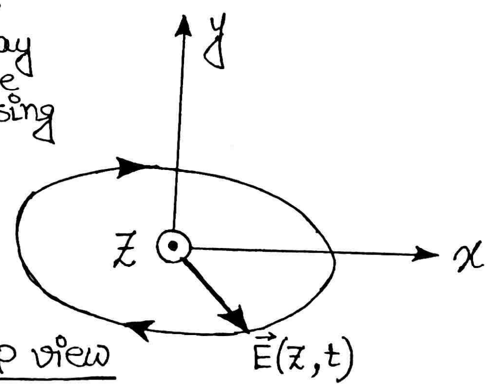

Side view Top view

Figure - 18.1 : Right elliptically polarized light (REP). wave propagating along z - axis; the $\vec{E}$ field is given by $\vec{E}(z,t)$ which is rotating about z-axis along an elliptical path in counter clock wise direction.

- Note : REP or RCP : Thumb of the right hand is along ray propagation direction and closing movement of the rest fingers gives rotation direction.




- In the similar way LEP or LCP can be represented by using Left hand.




side view top view

Figure - 18.2 : Left elliptically polarized light (LEP). Wave is propagating along z - direction. $\vec{E}$ field given by $\vec{E}(z,t)$. $\vec{E}$ field vector is rotating about z - axis along an elliptical path in clockwise direction.

- RCP and LCP can be represented this way.
- Note : In some books we may find that the conversion of rotation directions are reversed between REP and LEP also for RCP and LCP.

# 4. Production of linearly polarized light :

## Polarization by Dichoric crystal :

Dichoric crystal is a kind of anistropic (uniaxial) crystal. In such crystals, unpolarized light splits into two polarized components. One becomes polarized along the direction of optic axis (e-ray) and another polarized perpendicular to the optic axis and propagates as o-ray. Further, in dichoric crystal any one of these mutually perpendicular polarized component gets absorbed and the another propagates without significant absorption. Therefore, a dichoric crystal of sufficient thickness can produce plane polarized light. Tourmaline is an example of dichoric crystal.

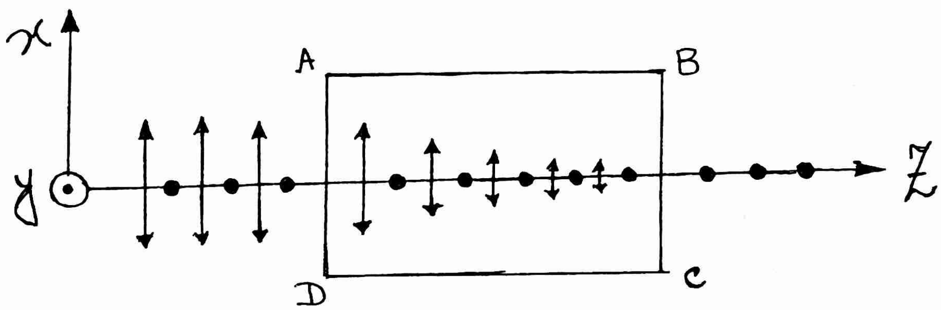
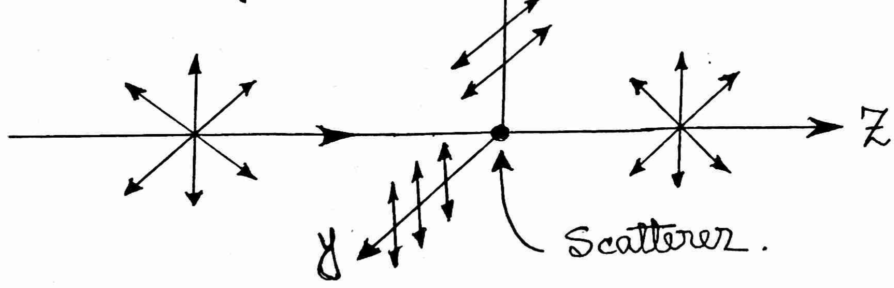



Figure - 19.1 : Two dimensional pictorial representation of dichorism. ABCD is representing a dichoric crystal. Unpolarized light propagating along z direction enters the crystal and the x-directed polarization is being absorbed strongly and y-directinal polarization remain almost unabsorbed. Finally, plane polarized EM wave produced with polarization along y-direction.

## Polarization by scattering :



The phenomena of polarization of light by dipole scattering is represented in the above diagram.

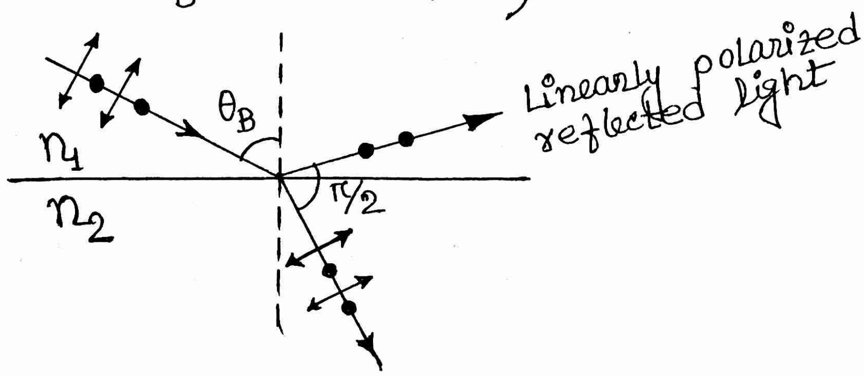
Consider unpolarized light is propagating along z-direction. A dipole scatterer in the path of the light will start oscillating in x-y plane due to influence of electric field of unpolarized EM wave. Light being transverse wave, light emitted from the oscillating dipole along perpendicular direction of incident light will become linearly polarized. Light propagating along x-axis is y-polarized and vice-versa. However, light propagating other than the plane of vibration (i.e; x-y plane in this example) remain un-polarized.

## Polarization by reflection :

If unpolarized light is reflected from the interface of two dielectric media, then for a particular angle of incidence (called Brewster's angle) the reflected light becomes plane polarized along the normal direction of the plane of incidence. Successive arrangment of these dielectrics can give rise to polarized transmitted light also. In transmitted light the polarization lies in the plane of incidence.

If $n_1$ and $n_2$ are the refractive indices of the medium from which light is coming and to which light is going respectively, then the Brewster's angle is given by;

$$\theta_B = \tan^{-1}(n_2/n_1)$$



## Wire grid polarizer :

This is artificially made to produce linearly polarized EM wave. In this device, thin copper wires are placed parallel to each other. If unpolarized wave passes through it then the electric field component parallel to the copper wires influences the free electrons of copper to flow along each wire. The energy of EM wave for this polarization is being absorbed by the electrons to gain their kinetic energy. On the other hand, no such process is observed for polarization direction perpendicular to the alignment of copper wires.

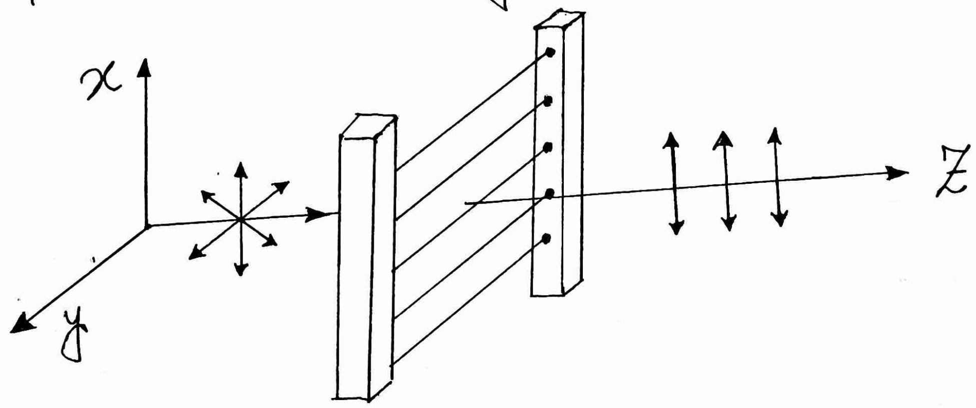


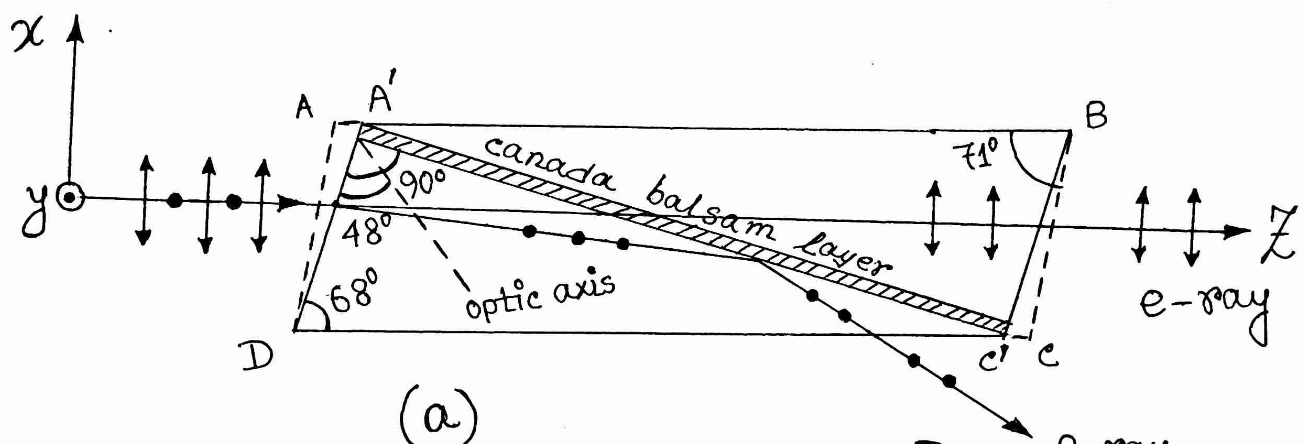

Figure - 21.1 : The wire grid polarizer. Copper wires are aligned along $y$-direction. Unpolarized light propagating along $z$-direction. After passing through the wire grid arrangement, the electric field vibration along $y$-direction gets absorbed and the emergent ray become linearly polarized along $x$-direction.

## Polaroid :

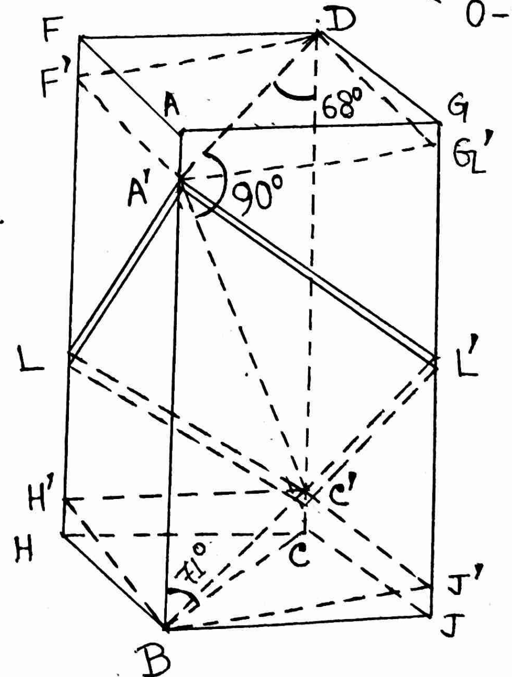
Polaroids are artificially made thin transparent film of long chain molecules aligned parallel to each other and the molecules contain atoms (e.g. iodine) which provide very high conductivity along the length of molecules. When EM wave passes through polaroids, due to high conductivity along the length of long chain molecules, the polaroids absorb electric field strength in this direction. The emergent wave becomes linearly polarized along the direction perpendicular to the length of long chain molecules of polaroids.

## Nicol prism :

Nicol prism is specially designed optical device made from calcite crystal. It is used to produce and analysis of polarized lights. Below we give the description and working mechanism of Nicol prism with suitable diagram.




(a)




(b)

(b) Describes the three dimensional geometry of Nicol prism. It was invented by William Nicol in 1828. This is the first man made device to produce polarized light.

Nicol prism made from rhombohedron structure of calcite crystal whose length was taken to be three times of its breadth.

The initial rhombhedron is DFAGBHCJ. The angle of refracting faces DFAG and

BHCJ with the adjacent edges is 71°. That means,

$$\angle ADC = \angle CBA = 71^\circ$$

The ADCB plane is one of principal section (parallel to optic axis).

Now the two refracting surfaces of the natural crystal is cut along the planes DF′A′G′ and BJ′C′H′ which reduces the acute angle of the principal section from 71° to 68°.

$$\angle A'DC' = \angle C'BA' = 68^\circ$$

Further the whole system is cut into two equal halves along the plane A′LC′L′ and these two halves are joined together with canada balsam layer.

The cutting plane A′LC′L′ and the principal section plane A′DC′B intercept each other along the line A′C′ such that

$$\angle DA'C' = \angle BC'A' = 90^\circ$$

Therefore, Nicol prism is combination of two identical parts joined together with transparent canada balsam layer. One part is DF′A′G′ A′LC′L′ bounded region and the second one is BH′C′J′C′L′A′L bounded region. They are pasted together at A′LC′L′ plane.

(a) Two dimensional representation of the principal section DA′BC′ (see figure (b)). The original crystal is DABC with 71° acute angle; from which the DA′BC′ is cut with 68° acute angle. A′C′ is canad balsam layer.

Calcite crystal is negative crystal, that means refractive index of ordinary light (say $n_o$) is larger than the smallest value of refractive index of extraordinary ray (say $n_e$). Note, only along the optic axis of ray propagation: both the refracting indices become equal. If two rays (o-ray and e-ray) propagat perpendicular to optic axis then the difference of refractive indices is maximum

For calcite crystal,

$$n_o = 1{\cdot}658 \quad , \quad n_e = 1{\cdot}486$$

and the refractive index of canada balsam is 1·526. That means canada balsam is optically rarer for o-ray and it is optically denser for e-ray while the light is coming from calcite medium.

Unpolarized light after entering into the Nicol prism from one of its refracting surfaces (say DF'A'G' in figure-(b) or equivalent to DA' in figure-(a)) splits into two components : o-ray and e-ray. At canada balsam layer, o-ray gets total internal reflection because for o-ray canada balsam layer is optically rarer. On the other hand e-ray passes through the canad balsam layer and emerges from the opposite refracting surface of Nicol prism. e-ray is polarized in the plane of principal section, so, the emergent ray (e-ray) becomes linearly polarized in the plane of principal section of Nicol prism.

## 5. Malus's Law : if plane polarized light transmits through polaroid then

This is experimentally obtained observation that intensity of linearly polarized light transmitted varies with the angle between passing axis of the polaroid and polarization direction of incident plane polarized light. The intensity variation is given by (of transmitted light),

$$I(\theta) = I_0 \cos^2\theta$$

where, $\theta$ = angle between plane of polarization of incident plane polarized light and the passing axis of polarizer through which it will transmit.

$I_0$ = Intensity of incident plane polarized light.

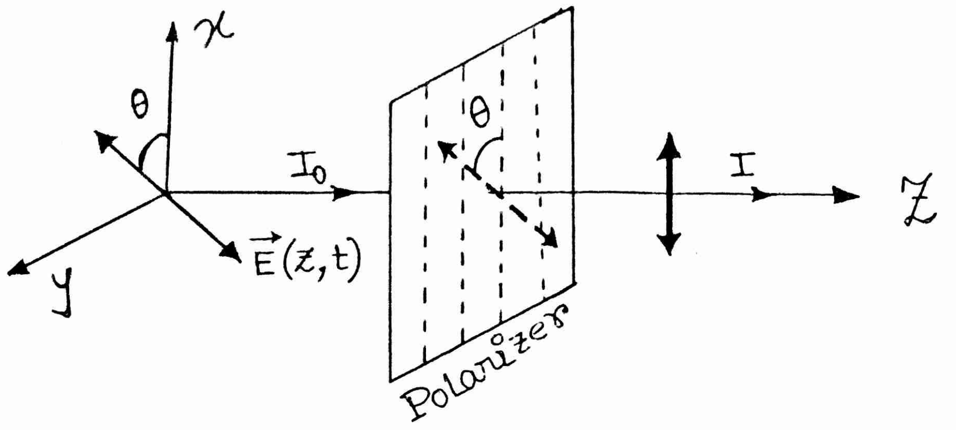



Figure - 25.1 : Linearly polarized light propagating along $z$-direction. Incident wave intensity is $I_0$ and polarization direction is making angle $\theta$ with $x$-axis. A polaroid is placed such that its passing axis is along $x$-axis. Ray coming out from polarizer will be $x$-polarized, that means polarization plane is rotated by angle $\theta$. From Malus's law, intensity of light coming out from polaroid will be,

$$I = I_0 \cos^2\theta .$$

● Note : If unpolarized light passes through a polaroid then the intensity of outgoing linearly polarized light becomes half of its original intensity. In unpolarized light, polarization direction continuously varies from 0 to $2\pi$ angle w.r.t the passing axis of polaroid. Therefore, intensity of outgoing plane polarized light will be given by,

$$I = I_0 \langle \cos^2\theta \rangle \quad ; \text{ average over } \theta \text{ from } 0 \text{ to } 2\pi$$

$$= I_0/2$$

**Problem - 26.1 :** An ellipse in a coordinate plane say $x-y$ is rotated about $z$-axis. The centre of the ellipse coincide with the origin of coordinate. Find the general form of such rotated ellipse.

**Solution :**

The rotated ellipse is shown in the adjacent figure.
$\tilde{x}$ and $\tilde{y}$ are the symmetry axes of the ellipse.
In these coordinate variables the form of ellipse will be;

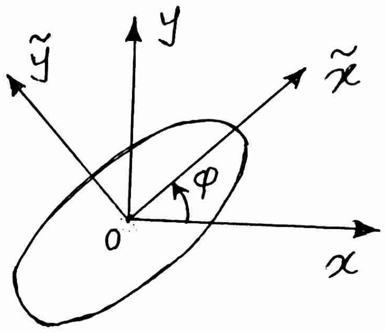



Figure - 26.1

$$\frac{\tilde{x}^2}{a^2} + \frac{\tilde{y}^2}{b^2} = 1 \qquad \text{——— (26.1)}$$

$\tilde{x}-\tilde{y}$ coordinate axes are obtaind by rotating $x-y$ axes by angle $\varphi$. Therefore,

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}}_{\text{Rotation matrix by angle } +\varphi} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{——— (26.2)}$$

Now substitute the transformation relations of (26.2) into (26.1);

$$\Rightarrow \quad \frac{1}{a^2}\left(x\cos\varphi + y\sin\varphi\right)^2 + \frac{1}{b^2}\left(-x\sin\varphi + y\cos\varphi\right)^2 = 1$$

It takes the general form of,

$$\boxed{Ax^2 + By^2 + Cxy + D = 0} \qquad \text{—— (26.3)}$$

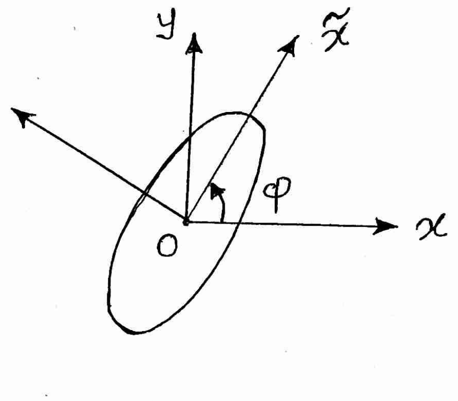
where, A, B, C and D are constants. This is the <u>equation of rotated ellipse</u>.

**Problem - 27.1** : The general form of rotated ellipse in $x-y$ plane about the origin is given by,

$$Ax^2 + By^2 + Cxy + D = 0$$

Find the angle of rotation in terms of the constants A, B, C and D.

**Solution** :




Suppose $\varphi$ be the rotation angle of the ellipse. It is obvious that if $\tilde{x}-\tilde{y}$ be the symmtry axis of the rotated ellipse then in transformed coordinate $(\tilde{x}-\tilde{y})$, the equation of ellipse will be,

$$\frac{\tilde{x}^2}{a^2} + \frac{\tilde{y}}{b^2} = 1$$

That means the given equation,

$$Ax^2 + By^2 + Cxy + D = 0 \qquad \text{——— } (27.1)$$

is to be transformed in $\tilde{x}, \tilde{y}$ in which there will be no product term of $\tilde{x}\tilde{y}$.

The transformation,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}}_{\text{Rotation matrix for rotation angle of } -\varphi.} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$$

Substituting the transformation into (27.1) ;

$$A(\tilde{x}\cos\varphi - \tilde{y}\sin\varphi)^2 + B(\tilde{x}\sin\varphi + \tilde{y}\cos\varphi)^2 + C(\tilde{x}\cos\varphi - \tilde{y}\sin\varphi)(\tilde{x}\sin\varphi + \tilde{y}\cos\varphi) + D = 0$$

Now we have to set the coefficient of $\tilde{x}\tilde{y}$ to be zero,

$$\Rightarrow -A\sin 2\varphi + B\sin 2\varphi + C\cos 2\varphi = 0$$

$$\Rightarrow \boxed{\tan 2\varphi = \frac{C}{A-B}}$$

Here we can find the rotation angle of ellipse.

## 6. Superposition of perpendicular polarized light :

In this section we are going to discuss the formation of elliptically and circularly polarized light due to superposition of two linearly polarized light of same frequency but differing by phase and further two waves are polarized perpendicular to each other.

Consider two monochromatic wave is propagating along $z$-axis. One of which is polarized along $x$-axis and another along $y$-axis. They are represented as,

(28.0)- $x$-polarized light : $\vec{E}_x(z,t) = \hat{x} E_1 \cos(kz - \omega t)$
$y$-polarized light : $\vec{E}_y(z,t) = \hat{y} E_2 \cos(kz - \omega t - \delta)$

Phase difference between two waves is $\delta$.

At any arbitrary $z$ plane (let at $z=0$ plane for simplicity) these two electic field vibrations would represent SHM in the form of,

$$\left.\begin{aligned} x(t) &= E_1 \cos(\omega t) \\ y(t) &= E_2 \cos(\omega t + \delta) \end{aligned}\right\} \quad \text{—— (28.1)}$$

These two mutually perpendicular vibrations would give rise to resultant variation of electric field at $z=0$ plane. The resultant is,

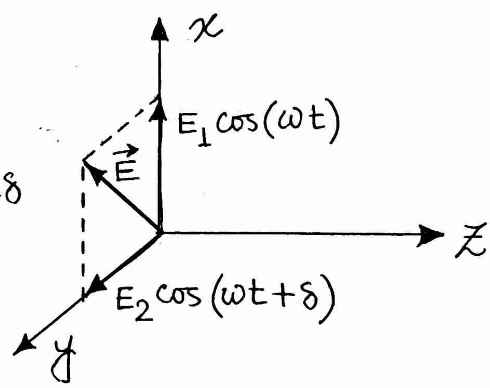
$$\vec{E}(0,t) = \hat{x} E_1 \cos(\omega t) + \hat{y} E_2 \cos(\omega t + \delta)$$

Now we have to find out the trajectory of the resultant.

$$y(t) = E_2 \cos(\omega t + \delta)$$

$$\Rightarrow \left(\frac{y}{E_2}\right) = \cos(\omega t)\cos\delta - \sin(\omega t)\sin\delta$$

$$= \left(\frac{x}{E_1}\right)\cos\delta - \left(1 - \cos^2(\omega t)\right)^{1/2}\sin\delta$$




$$\Rightarrow \Big[1 - \underbrace{\left(x/E_1\right)^2}_{\cos(\omega t)}\Big]^{1/2} \sin\delta = -\left(\frac{y}{E_2}\right) + \left(\frac{x}{E_1}\right)\cos\delta$$

Squaring both sides we get,

$$\left[1-\left(\frac{x}{E_1}\right)^2\right]\sin^2\delta = \left(\frac{y}{E_2}\right)^2 + \left(\frac{x}{E_1}\right)^2\cos^2\delta - \frac{2xy}{E_1E_2}\cos\delta$$

$$\Rightarrow \boxed{\frac{x^2}{E_1^2} + \frac{y^2}{E_2^2} - \frac{2xy}{E_1E_2}\cos\delta = \sin^2\delta} \quad \text{——— } (29\cdot1)$$

This relation represent the equation of ellipse (rotated)

$$Ax^2 + By^2 + Cxy + D = 0 \quad (\text{see problem}-26\cdot1)$$

where,

$$A = 1/E_1^2, \quad B = 1/E_2^2, \quad C = -\frac{2\cos\delta}{E_1E_2}$$

$$D = -\sin^2\delta.$$

The rotation angle;

From problem-27·1

$$\varphi = \frac{1}{2}\tan^{-1}\left(\frac{C}{A-B}\right) = \frac{1}{2}\tan^{-1}\left(\frac{-\frac{2\cos\delta}{E_1E_2}}{\frac{1}{E_1^2}-\frac{1}{E_2^2}}\right)$$

$$\Rightarrow \boxed{\varphi = \frac{1}{2}\tan^{-1}\left(\frac{2E_1E_2\cos\delta}{E_1^2-E_2^2}\right)} \quad \text{——— } (29\cdot2)$$

With the help of these two relations (29·1) and (29·2) we can find out the shape of the elliptically polarized light formed due to superposition of two mutually perpendicular plane polarized light of same frequency with difference in phase (given by (28·0)).

- According to (28·1), variation of $x$ is $\pm E_1$ and variation of $y$ is $\pm E_2$. That means the ellipse of (29·1) will be bounded by the rectangle of $x$-length $-E_1$ to $+E_1$ and of $y$-length $-E_2$ to $+E_2$.

## Special cases :

Here we represent the resultant polarization due to superposition of linear polarization (perpendicular to each other)

- Let us consider; $E_1 \approx E_2$ and $E_1$ slightly greater than $E_2$. Assume $E_1 \approx E_2 = E_0$.

Then (29.1) becomes;

$$x^2 + y^2 - 2xy \cos\delta = E_0^2 \sin^2\delta \quad ; \quad E_1 \approx E_2 = E_0$$

and (29.2) becomes;

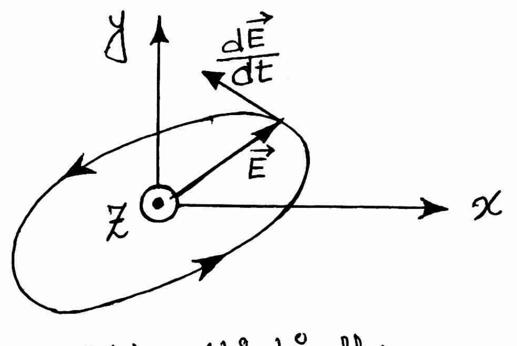
$$\varphi = \frac{1}{2}\tan^{-1}\left(\frac{2E_0^2 \cos\delta}{\epsilon}\right) \quad ; \quad \epsilon = (E_1^2 - E_2^2) = \text{very small number.}$$

With the help of these two relations we find the trajectories of the resultant elliptical polarization. But it is not obvious that whether it is left elliptically polarized or right elliptically polarized (REP).

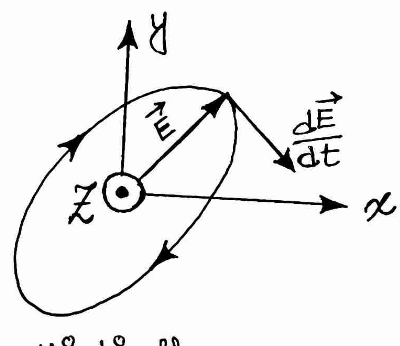
- If we have REP state in $x$-$y$ plane then from the diagram shown below it is obvious that

$$\vec{E} \times \frac{d\vec{E}}{dt}$$

will be along $z$-axis and for LEP the vector cross product will be directed along $-ve$ $z$-axis.




Right elliptically polarized (REP)
$\vec{E} \times \frac{d\vec{E}}{dt}$ is along $+ve$ $z$-axis




Left elliptically polarized (LEP)
$\vec{E} \times \frac{d\vec{E}}{dt}$ is directed along $-ve$ $z$-axis

In this considered case at $z = 0$ plane,

$$\vec{E}(t) = \hat{x}\, E_1 \cos(\omega t) + \hat{y}\, E_2 \cos(\omega t + \delta)$$

$$\Rightarrow \quad \frac{d\vec{E}}{dt} = -\omega\left[\hat{x}\, E_1 \sin(\omega t) + \hat{y}\, E_2 \sin(\omega t + \delta)\right].$$

$$\Rightarrow \quad \vec{E} \times \frac{d\vec{E}}{dt} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ E_1 \cos(\omega t) & E_2 \cos(\omega t + \delta) & 0 \\ -\omega E_1 \sin(\omega t) & -\omega E_2 \sin(\omega t + \delta) & 0 \end{vmatrix}$$

$$= \omega E_1 E_2 \left[\sin(\omega t)\cos(\omega t + \delta) - \sin(\omega t + \delta)\cos(\omega t)\right]$$

$$\Rightarrow \boxed{\vec{E} \times \frac{d\vec{E}}{dt} = -\omega \underbrace{E_1 E_2}_{\approx E_0^2} \sin\delta \approx -\omega E_0^2 \sin\delta}$$

The vector cross product is +ve when $\pi < \delta < 2\pi$ and -ve when $0 < \delta < \pi$.

- **Finally we have;**

The linear polarizations:
$$\begin{cases} x(t) = E_1 \cos(\omega t) \approx E_0 \cos(\omega t) \\ y(t) = E_2 \cos(\omega t + \delta) \approx E_0 \cos(\omega t + \delta) \end{cases}$$

to produce ellipse,

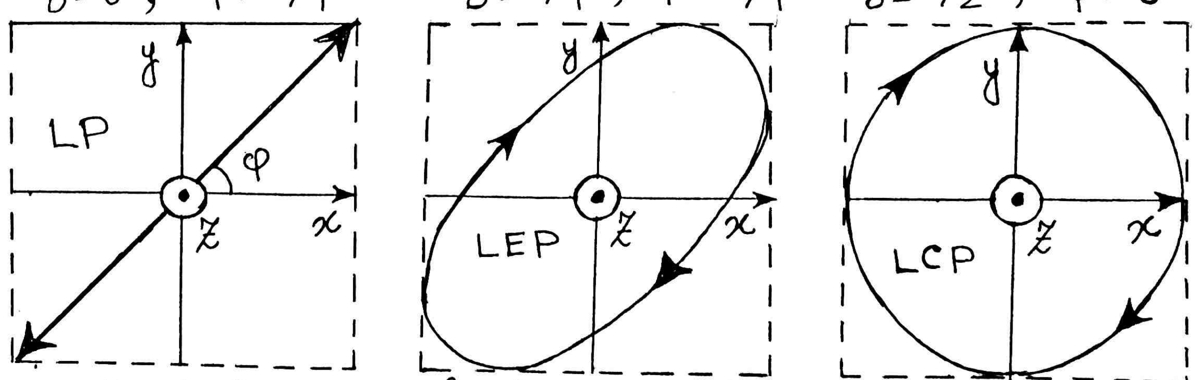
$$x^2 + y^2 - 2xy\cos\delta = E_0^2 \sin^2\delta$$

with rotation angle,

$$\varphi = \frac{1}{2}\tan^{-1}\left(\frac{2E_0^2\cos\delta}{\epsilon}\right) \; ; \; \epsilon = (E_1^2 - E_2^2) \gtrsim 0$$

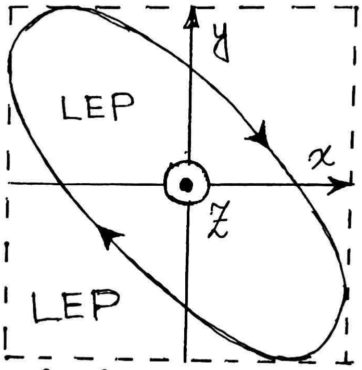
LEP for $0 < \delta < \pi$ and REP for $\pi < \delta < 2\pi$.

$\delta = 0$, $\varphi = \pi/4$ $\qquad$ $\delta = \pi/4$, $\varphi = \pi/4$ $\qquad$ $\delta = \pi/2$, $\varphi = 0$



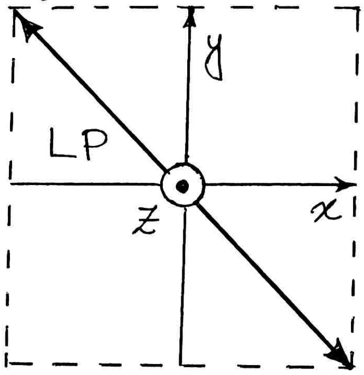

$y - x = 0$ $\qquad$ $x^2 + y^2 - \sqrt{2}\,xy = E_0^2/2$ $\qquad$ $x^2 + y^2 = E_0^2$

$\delta = 3\pi/4$, $\phi = -\pi/4$



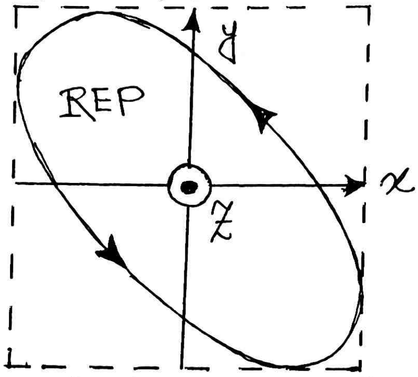

$x^2 + y^2 + \sqrt{2}\,xy = E_0^2/2$

$\delta = \pi$, $\varphi = -\pi/4$



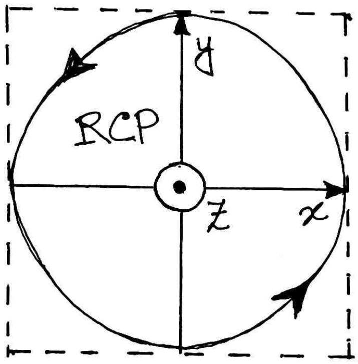

$x + y = 0$

$\delta = 5\pi/4$, $\varphi = -\pi/4$



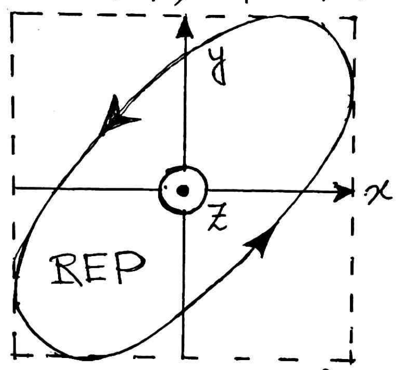

$x^2 + y^2 + \sqrt{2}\,xy = E_0^2/2$

$\delta = 3\pi/2$, $\varphi = 0$



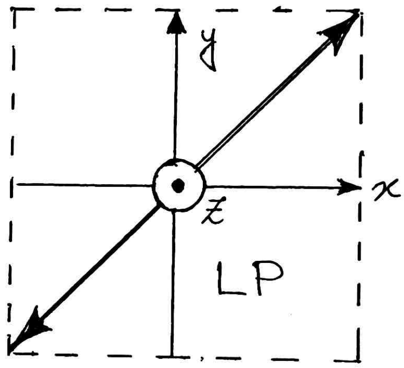

$x^2 + y^2 = E_0^2$

$\delta = 7\pi/4$, $\varphi = \pi/4$




$x^2 + y^2 - \sqrt{2}\,xy = E_0^2/2$

$\delta = 2\pi$, $\varphi = \pi/4$

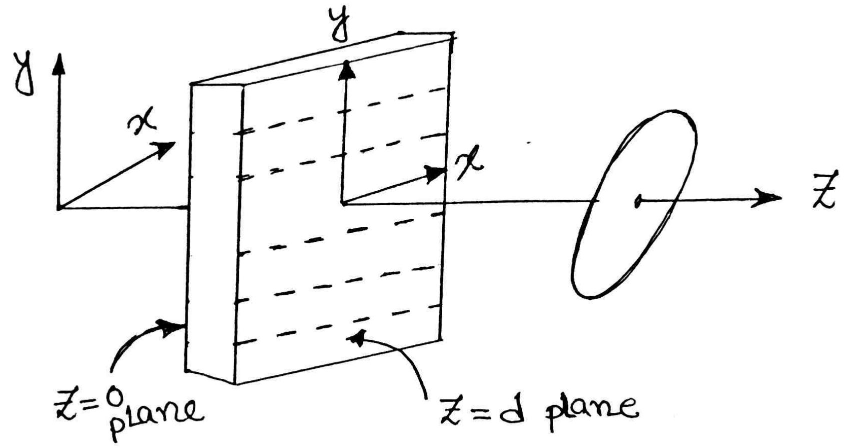



$x - y = 0$

## 7. Retardation plates :

Retardation plate is thin plate made of uniaxial crystal (doubly refracting crystal) such that the optic axis remain parallel to the refracting faces of the plate.

- If any light ray falls on retardation plate with an arbitrary polarization state then it splits into two polarization componenets inside the retardation plate. One component is polarized along the optic axis (propagates as e-ray) and another along perpendicular to optic axis (propagates as o-ray).




Suppose light entering the retardation plate at $z=0$ plane and the exit plane is $z=d$ plane. Further we consider that the optic axis of the plate is along $x$-axis.

The e-ray is polarized along optic axis (here $x$-axis) So, the o-ray will be polarized along $y$-axis (perpendicular to optic axis).

O-ray and e-ray travell with different speed through uniaxial crystal. Therefore, in the retardation plate, there will be phase difference between o-ray and e-ray at exit and since they are perpendicular to each other they will superpose to give different states of polarization (discussed in section-6).

Let,

$E_1$ = Amplitude of electric field for e-ray.

$E_2$ = Amplitude of electric field for o-ray.

Then at $z=d$ (exit surface of retardation plate);

$$\left.\begin{aligned} \text{For e-ray} &: \quad E_x(t) = E_1 \cos(K_e d - \omega t) \\ \text{For O-ray} &: \quad E_y(t) = E_2 \cos(K_o d - \omega t) \end{aligned}\right\} \quad \text{—(33.1)}$$

$$\left.\begin{aligned} K_o &= \frac{\omega}{v_o} = \frac{\omega n_o}{c} \\ K_e &= \frac{\omega}{v_e} = \frac{\omega n_e}{c} \end{aligned}\right\}$$

'o' stands for ordinary

'e' stands for extraordinary.

$$E_y(t) = E_2 \cos(K_0 d - \omega t)$$

$$\Rightarrow \frac{E_y}{E_2} = \cos\left(\frac{\omega n_o d}{c} - \omega t\right) = \cos\left(\underbrace{\frac{\omega n_e d}{c} - \omega t}_{\tau} + \underbrace{\frac{\omega d}{c}(n_o - n_e)}_{\delta}\right)$$

$$\Rightarrow \frac{E_y}{E_2} = \cos(\tau + \delta)$$

where, $\tau = \left(\frac{\omega n_e d}{c} - \omega t\right)$, $\delta = \frac{\omega d}{c}(n_o - n_e)$

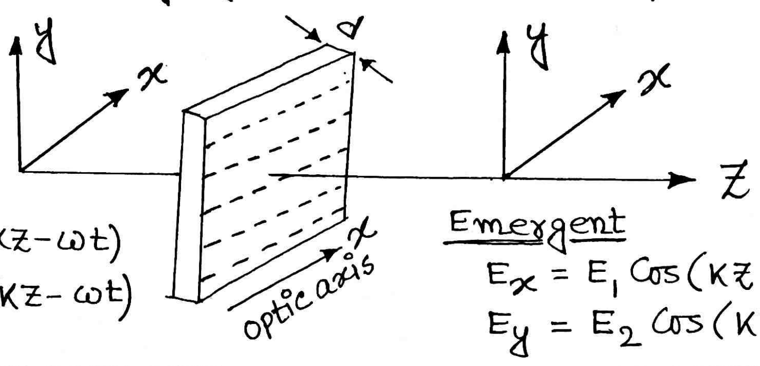
$$\frac{E_y}{E_2} = \cos(\tau + \delta)$$
$$= \cos\tau \cos\delta - \sin\tau \sin\delta$$
$$= \underbrace{\left(\frac{E_x}{E_1}\right)}_{\cos(\tau) \text{ from } (33 \cdot 1)} \cos\delta - (1 - \cos^2\tau)^{1/2} \sin\delta$$
$$= \frac{E_x}{E_1} \cos\delta - \left[1 - \left(E_x/E_1\right)^2\right]^{1/2} \sin\delta$$

$$\Rightarrow E_y^2/E_2^2 + \left(E_x^2/E_1^2\right)\cos^2\delta - \frac{2E_x E_y}{E_1 E_2}\cos\delta = \left[1 - \left(\frac{E_x}{E_1}\right)^2\right]\sin^2\delta$$

$$\Rightarrow \frac{E_x^2}{E_1^2} + \frac{E_y^2}{E_2^2} - \frac{2E_x E_y}{E_1 E_2}\cos\delta = \sin^2\delta$$

That means the emergent ray from retardation plate will be elliptically polarized if two perpendicular components are given by (33·1).

**Note :** After passing throug retardation plate two rays o-ray and e-ray gain some relative phase shift.




Incident
$E_x = E_1 \cos(Kz - \omega t)$
$E_y = E_2 \cos(Kz - \omega t)$

Emergent
$E_x = E_1 \cos(Kz - \omega t)$
$E_y = E_2 \cos(Kz - \omega t + \delta)$

$$\underbrace{\delta = \frac{\omega d}{c}(n_o - n_e)}$$

## Quarter wave plate :

It is a retardation plate which introduces $\pi/2$ phase difference between o-ray and e-ray.
If we consider -ve crystal ($n_o > n_e$) then,

$$\delta = \frac{\omega d}{c}(n_o - n_e) = \pi/2$$

$$\Rightarrow d = \frac{c}{\omega}(n_o - n_e)\,\pi/2$$

$$\Rightarrow \boxed{d = \frac{\lambda}{4}(n_o - n_e)} \quad ; \quad \omega = 2\pi\nu = \frac{2\pi c}{\lambda}$$

Thickness of quater wave plate

## Half wave plate :

This retardation plate introduces $\pi$ phase difference between o-ray and e-ray.
For -ve crystal ($n_o > n_e$);

$$\delta = \frac{\omega d}{c}(n_o - n_e) = \pi.$$

$$\Rightarrow \boxed{d = \frac{\lambda}{2}(n_o - n_e)}$$

Thickness of half wave plate.

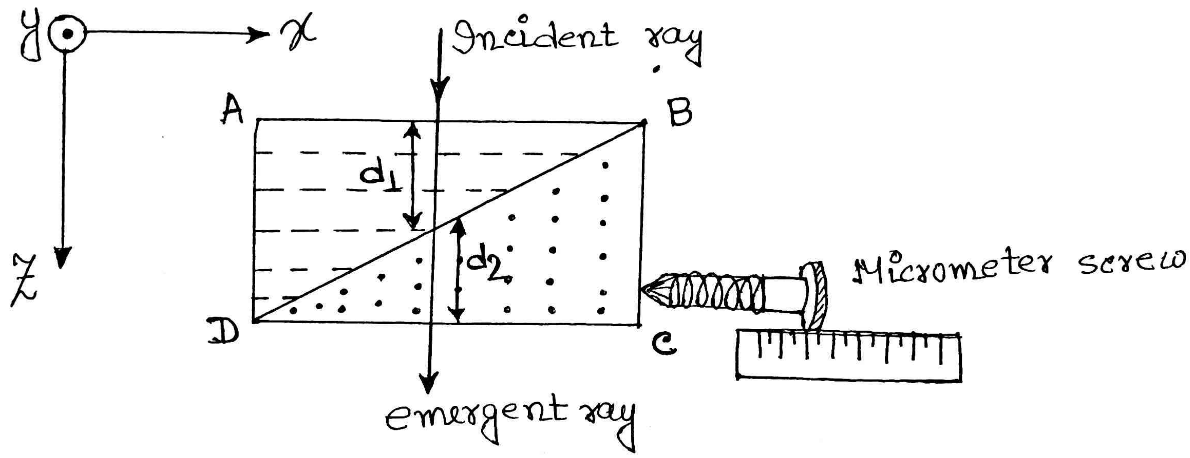
## 7.1 Babinet's compensator :

For a given quarter wave plate (QWP) or half wave plate (HWP) its thickness d and $(n_o - n_e)$ being fixed it is useful to work for a particular wave length $\lambda$ and in the very close regime of $\lambda$. The relations are given by;

$$d = \frac{\lambda}{4}(n_o - n_e) \; ; \text{ for QWP}$$

$$d = \frac{\lambda}{2}(n_o - n_e) \; ; \text{ for HWP.}$$

In the contrary, Babinet's compensator is an variable retardation plate. It can be useful for a wide range of $\lambda$ to incorporate desired phase shift between o-ray and e-ray. Below, the structure of Babinet's compensator is ~~shown~~ described with its working mechanism.




Babinet's compensator consists of two thin right angled prism of uniaxial crystal. One is ABD and another is BCD. They are joined together at the hypotenuse. Optic axis of one prism is aligned perpendicular to that of another. Light ray allowed to pass through the combination such that ray propagation direction remain perpendicular to both optic axes. In the diagram prism ABD has optic axis parallel to $x$-axis, prism CDB has it parallel to $y$-axis and light ray is propagating along $z$-axis.

Due to mutual perpendicular alignment of optic axis, the role of e-ray and o-ray is reversed in two parts as;

For part ABD :

e-ray is $x$- polarized (parallel to optic axis)
o-ray is $y$- polarized (perpendicular to optic axis)

For part BCD :

Parallel to opticaxis : e-ray is $y$-polarized
Perpendicular " : o-ray is $x$- polarized.

That means relative phase shift can be tuned between o-ray and e-ray by tuning the ~~length the~~ path length of light ray passing through part ABD and part BCD.

Consider light ray is passing through near the central region of the device.

If $d_1$ and $d_2$ be the thickness of two parts of Babinet's compensator through which light ray is passing. Then relative phase shift of o-ray and e-ray is,

$$\delta = \frac{\omega d_1}{c}(n_o - n_e) - \frac{\omega d_2}{c}(n_o - n_e)$$

-ve sign is due to reversal of o-ray and e-ray from part ABD to part BCD.

$$\Rightarrow \boxed{\delta = \frac{\omega}{c}(d_1 - d_2)(n_o - n_e)}$$

With the help of the micrometer screw the prism BCD can be shifted forward and backward direction which in effect produces variable value of $d_2$. Thus the phase shift $\delta$ between o-ray and e-ray can be tuned.

If $l$ be the minimum lateral shift of micrometer screw such that same state of polarization is brought back of the emergent ray then we can say that,

$l$ amount lateral shift is equivalent to $2\pi$ phase shift.

⇒ To produce $\varphi$ phase shift the required movement of micrometer screw will be,

$$\boxed{x = \frac{\varphi l}{2\pi}}$$

# Polarization of Light (Part - III)

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April, 2022

## Contents

# References :


1. Optics; Ajoy Ghatak; McGraw Hill
2. Introduction to Electrodynamics; D. J. Griffiths; Prentice Hall


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## 1 Analysis of Polarized Light

Light can be characterized by different states of polarization, which may be any one of the following states,

1. Unpolarized (UP)
2. Linearly polarized (LP)
3. Circularly polarized (CP)
4. Elliptically polarized (EP)
5. Linearly polarized + Unpolarized (LP+UP)
6. Circularly polarized + Unpolarized (CP+UP)
7. Elliptically polarized + Unpolarized (EP+UP)

With the help of polarizer (e.g. Nicol prism) and $\lambda/4$ (quarter wave) plate we can analysis the state of polarization of light. Following we discuss how to detect the state of polarization state of light.

First, place a polarizer in the path of light. Rotate the polarizer gradually by $360^0$ about the direction of propagation and observe the intensity of light passing through the polarizer. Any one of the following observations,

Observation - 1
*or*, Observation - 2
*or*, Observation - 3

may be recorded and we can conclude about the observations as discussed below.

- **Observation - 1 :** Complete extinction of light intensity at two orientation (separated by $180^0$) of polarizer.
  **Conclusions :** The incident light is linearly polarized.
  **Explanation :** When the passing axis becomes perpendicular to the plane of vibration, no light intensity passes through the polarizer.

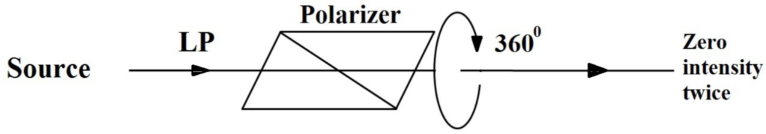


Figure 1:

- **Observation - 2 :** No variation of intensity observed over the full rotation of polarizer.
  **Conclusions :** Incident light can be in any one of the following state,
  (i) Unpolarized
  or
  (ii) Circularly polarized
  or
  (iii) Unpolarized + Circularly polarized

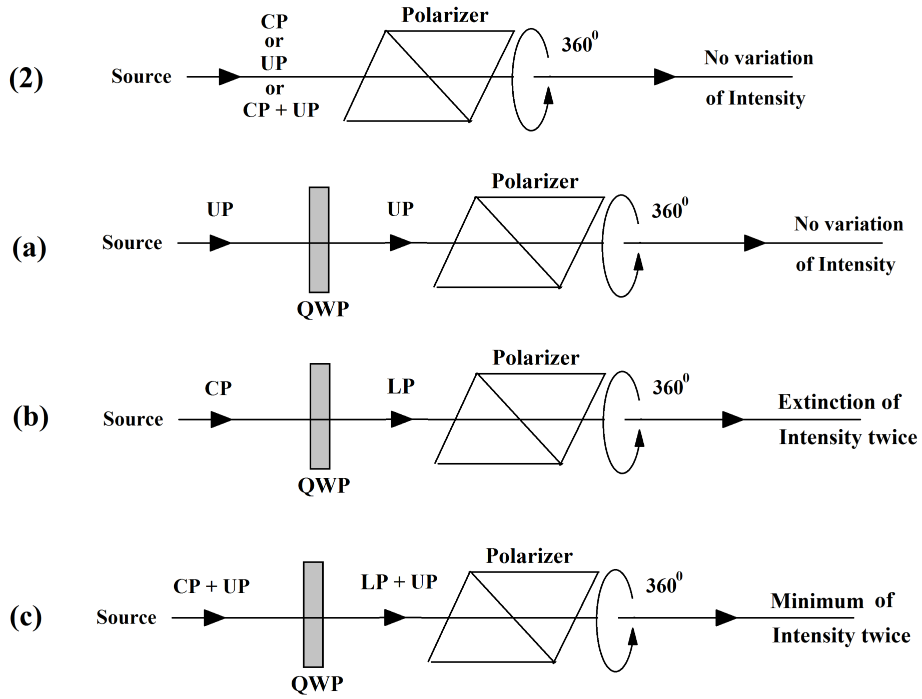


Figure 2:

Now we have to determine specifically the state of polarization out of these three possibilities.

For this purpose, place a $\lambda/4$ plate in the path of light in between source and polarizer. Then, rotate the polarizer gradually by $360^0$ and observe the intensity. Any one of the following observations can be recorded,

- *Observation - 2(a) :* No variation of intensity over full rotation of polarizer.
  *Conclusion :* Light beam is Unpolarized.
- *Observation - 2(b) :* Complete extinction of light intensity at two different rotation angle of polarizer.
  *Conclusion :* Light beam is Circularly polarized.
  *Explanation :* Quarter wave plate transforms circularly polarized light into plane polarized light. After passing through polarizer, intensity of plane polarized light becomes zero when the plane of polarization and passing axis of polarizer are mutually perpendicular.
- *Observation - 2(c) :* Variation of intensity observed (without complete extinction) over full rotation of polarizer.
  *Conclusion :* The light beam is Circularly polarized + Unpolarized.

- **Observation - 3 :** Variation of intensity of light observed but without complete extinction of intensity over the full rotation of polarizer.
  **Conclusions :** Incident light can be in any one of the following state,
  (i) Elliptically polarized
  or
  (ii) Elliptically polarized + Unpolarized
  or
  (iii) Linearly polarized + Unpolarized

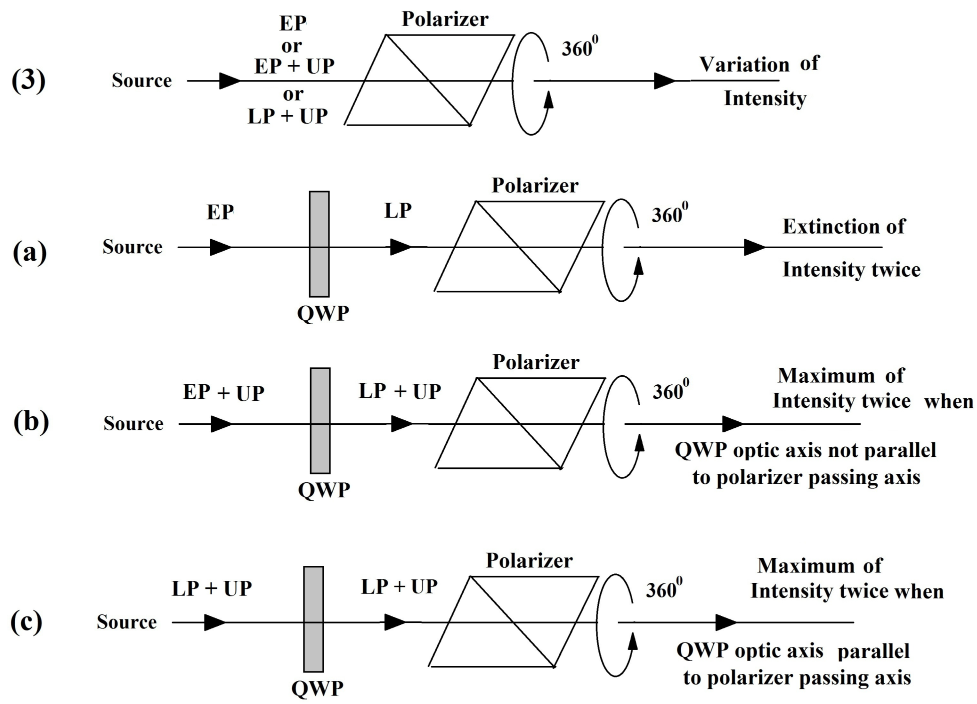


Figure 3:

Now we have to determine specifically the state of polarization out of these three possibilities.

For this purpose, set the polarizer for maximum intensity and place a $\lambda/4$ plate in between source and polarizer such that optic axis of the plate is kept parallel to passing axis of polarizer. Then, rotate the polarizer gradually by $360^0$ and observe the light intensity. Any one of the following observations can be recorded.

– *Observation - 3(a) :* Complete extinction of light intensity for two different rotation angle of polarizer.
  *Conclusions :* The light is Elliptically polarized.
  *Explanation :* Quarter wave plate converts elliptically polarized light into plane polarized light in which plane of polarization makes an angle with the optic axis of the plate. So, when this converted LP light passes through the polarizer, it is extinguished for perpendicular orientation of passing axis with the plane of polarization.

– *Observation - 3(b) :* By rotating the polarizer, variation of intensity without complete extinction can be observed. Maximum intensity is observed when optic axis of quarter wave plate is inclined with the passing axis of polarizer.
  *Conclusion :* The light is Elliptically polarized + Unpolarized.
  *Explanation :* Same of the above (explanation for observation - 3(a)). Further, the non vanishing intensity is due to presence of unpolarized light.

– *Observation - 3(c) :* Variation of intensity without complete extinction. Maximum intensity is observed when optic axis of quarter wave plate is parallel with the passing axis of polarizer.
  *Conclusion :* The light is Linearly polarized + Unpolarized.
  *Explanation :* Linearly polarized light is converted to elliptically polarized light after passing through the quarter wave plate. The axes of the ellipse are aligned to optic axis and perpendicular to optic axis of the quarter wave plate. So, when polarizer passing axis becomes parallel to the optic axis (i.e. also the axes of ellipse), we obtain maximum intensity. The non vanishing intensity is due to the effect of unpolarized light.

# 2 Optical activity

• In broad sense, *optical activity* or *rotatory polarization* is the phenomena in which, the plane of polarization of a plane polarized light rotates continuously during the propagation of light along the optic axis of some kind of crystals (such as quartz) or during propagation of light through solution of some kind of substances (such as water-sugar solution). The substances in which this phenomena is observed are called *optically active* substances.
• The origin of optical activity is molecular structure of substances in solution. Due to helical structure of the sugar molecule, it shows optical activity in water solution.
• In case of crystals, structural asymmetry about optic axis gives rise to optical activity.
• The plane of polarization of light in optically active medium can be rotated towards right or

left depending on the atomic arrangement of substance. The former one is called *dextro-rotatory* and the latter is called *laevo-rotatory.*

## 2.1 Biot's law

Biot gave some important laws on optical activity after performing systematic observation on optical activity. The laws are discussed in the following.

1. The rotation angle of the polarization plane of plane polarized light produced by optically active substance is directly proportional to the length traversed by the light in the substance.

2. The combined rotation produced by two different substances having different thickness is algebraic sum of the rotations produced by individual substances separately.

3. In case of solution of optically active substances, the rotation angle of polarization plane is directly proportional to concentration of the substances in the solution.

4. The rotation angle of polarization plane depends on the wavelength of light and temperature. The rotation is approximately proportional to the inverse square of wave length.

• **Consider a solution of optically active substance.**
Let,

$\theta =$ Rotation angle of polarization plane

$l =$ Length of solution through which light is passing

$m =$ Mass of optically active substance per unit volume of solution

Then, from Biot's laws,

$$\theta \propto l \times m$$

or,

$$\theta = slm$$

where, the constant $s$ is called **specific rotation**. It depends on the active substance and temperature.

The specific rotation for active solution is defined as :

> *The amount of rotation of polarization plane produced by optically active solution of length 10 cm containing 1 gm of optically active substance per 1 cc of solution.*

• **Note :** In the above we have defined the specific rotation for active solution. In case of crystal it is defined as,

> *The amount of rotation of polarization plane produced by optically active crystal of thickness 1 mm..*

• **Rotatory dispersion :** According to Biot's laws, rotation angle of polarization plane by active medium is approximately proportional to the inverse square of the wavelength. So, if plane polarized white light passes through optically active medium, then, polarization plane of different wavelength components rotate by different angles. This will give rise to angular variation of colours in the sequence of red, yellow, green, blue, violet (in the increasing order of rotation angle).

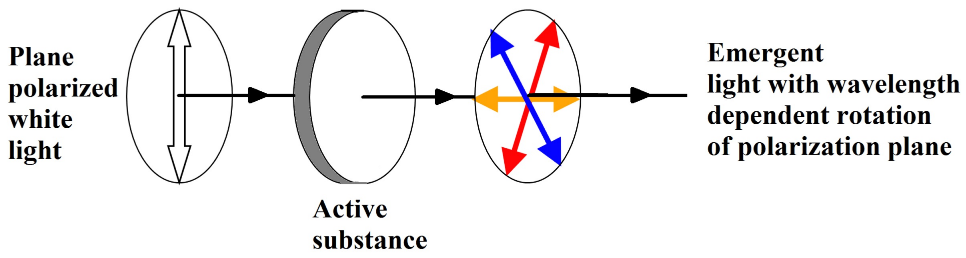


Figure 4: Schematic representation of rotatory dispersion.

## 2.2 Fresnel's explanation of rotatory optical activity

According to Fresnel, plane polarized light can be considered as superposition of two opposite circularly polarized light (one is right circularly polarized light (**RCP**) and another is left circularly polarized light (**LCP**)) of equal amplitude. In optically active medium, right circularly polarized light (RCP) and left circularly polarized light (LCP) propagate with two different speed, therefore, they gain relative phase difference. This phase difference increases with traveling distance. This continuously increasing phase shift between RCP and LCP leads to continuous rotation of polarization plane of incident plane polarized light. Following we develop the mathematical theory of Fresnel's explanation on optical activity.

### 2.2.1 Mathematical treatment of Fresnel's theory

Suppose plane polarized light is propagating along $z$-direction in empty space. We consider the polarization is along $x$-direction. We can verify that the plane polarized light can be considered as superposition of RCP and LCP.
In empty space, both RCP and LCP propagates with same speed. So, they have same value of wave vector. Now, in empty space components of electric field are represented as,

For RCP :

$$x \text{ component : } \quad E_{r,x} = E_0 \cos(kz - \omega t)$$

$$y \text{ component : } \quad E_{r,y} = E_0 \cos(kz - \omega t + \pi/2)$$

For LCP :

$$x \text{ component : } \quad E_{l,x} = E_0 \cos(kz - \omega t)$$

$$y \text{ component : } \quad E_{l,y} = E_0 \cos(kz - \omega t - \pi/2)$$

Therefore, superposition of these RCP and LCP in empty space gives resultants as,

$$E_x = (E_{r,x} + E_{l,x}) = 2E_0 \cos(kz - \omega t)$$

$$\text{and} \quad E_y = (E_{r,y} + E_{l,y}) = 0$$

So, it is verified that the incident plane polarized light is $x$-polarized light.
Now, after entering into an active medium, RCP and LCP will propagate with different speed. So, RCP and LCP will have different values of wave vectors.
The electric field components of right circularly polarized light (RCP) can be given by,

$$x \text{ component : } \quad E_{r,x} = E_0 \cos(k_r z - \omega t)$$

$$y \text{ component : } \quad E_{r,y} = E_0 \cos(k_r z - \omega t + \pi/2)$$

Electric field components of left circularly polarized (LCP) light can be given by,

$$x \text{ component : } \quad E_{l,x} = E_0 \cos(k_l z - \omega t)$$

$$y \text{ component : } \quad E_{l,y} = E_0 \cos(k_l z - \omega t - \pi/2)$$

where,
$k_r$ = wave vector magnitude of RCP light
and $k_l$ = wave vector magnitude of LCP light.
That means,

$$k_r = \frac{\omega}{v_r} = \frac{\omega n_r}{c}$$

where, $v_r$ = speed of right RCP light and $n_r$ is the refractive index of the same.
Similarly, for LCP,

$$k_l = \frac{\omega}{v_l} = \frac{\omega n_l}{c}$$

where, $v_l$ = speed of right LCP light and $n_l$ is the refractive index of the same.
Now the resultant fields due superposition of LCP and RCP would give,

$$x \text{ component : } \quad E_x = E_{r,x} + E_{l,x} = E_0 \cos(k_r z - \omega t) + E_0 \cos(k_l z - \omega t)$$

or,

$$E_x = 2E_0 \cos\left(\frac{1}{2}(k_r - k_l)z\right) \cos\left(\frac{1}{2}(k_r + k_l) - \omega t\right)$$

Similarly for $y$ component,

$$E_y = E_{r,y} + E_{l,y} = E_0 \cos(k_r z - \omega t + \pi/2) + E_0 \cos(k_l z - \omega t - \pi/2)$$

or,

$$E_y = 2E_0 \cos\left(\frac{1}{2}(k_r - k_l)z + \pi/2\right) \cos\left(\frac{1}{2}(k_r + k_l) - \omega t\right)$$

or,

$$E_y = \; -2E_0 \sin\left(\frac{1}{2}(k_r - k_l)z\right)\cos\left(\frac{1}{2}(k_r + k_l) - \omega t\right)$$

Finally we have the resultant electric field due to superposition of LCP and RCP as,

$$x - \text{component}: \quad E_x = 2E_0 \cos\left(\frac{1}{2}(k_r - k_l)z\right)\cos\left(\frac{1}{2}(k_r + k_l) - \omega t\right)$$
$$y - \text{component}: \quad E_y = \; -2E_0 \sin\left(\frac{1}{2}(k_r - k_l)z\right)\cos\left(\frac{1}{2}(k_r + k_l) - \omega t\right)$$

The term representing wave propagation for two components is,

$$\cos\left(\frac{1}{2}(k_r + k_l) - \omega t\right)$$

Clearly, two mutually perpendicular components share same wave propagation part without any relative phase difference. That means the superposition of the components $E_x$ and $E_y$ will give plane polarized light. Because, we have already obtained that superposition of two mutually perpendicular vibrations without any phase difference lead to linear vibration inclined with coordinate axes.
Suppose, $\phi$ is the inclination of linear vibration of resultant $\boldsymbol{E}$ field with $x$ axis, then,

$$\tan\phi = \frac{E_y}{E_x} = \; -\tan\left(\frac{1}{2}(k_r - k_l)z\right)$$

According to our consideration, the incident light was $x$-polarized. Therefore, rotation angle of polarization plane in active medium is,

$$\boxed{\phi = \frac{1}{2}(k_l - k_r)z}$$

From this expression it is obvious that polarization plane rotates continuously as the plane polarized light propagates through active medium along $z$-direction.

- The active substance is said to be dextro-rotatory or right handed if the rotation angle $\phi$ is $+ve$. That means, the rotation appears to be counterclockwise when viewed to the source through active medium. On the other hand, active substance is said to be laevo-rotatory or left handed if the rotation angle $\phi$ is $-ve$. That means, the rotation appears to be clockwise when viewed to the source through active medium.

# 3 Faraday rotation

If a plane polarized light is propagating through a medium in presence of magnetic field along the direction of propagation of light, then, the polarization plane of the plane polarized light gets rotated. This phenomena is known as **Faraday rotation**. It was first discovered by Michael Faraday in 1845.
Rotation angle of polarization plane is given by,

$$\boxed{\theta = VHl} \tag{1}$$

where, $H$ is applied magnetic field intensity, $l$ is the length traveled by light in the medium in presence of magnetic field and $V$ is a constant called **Verdet constant**. The constant $V$ depends on the choice of medium. For silica, $V = 2.64 \times 10^{-4}$ degree/Ampere.

Faraday rotation effect is used to measure very high current ($\sim$ $10^3$ Ampere). Consider winding of single mode optical fiber over a current carrying conductor as shown in the figure-5. Plane

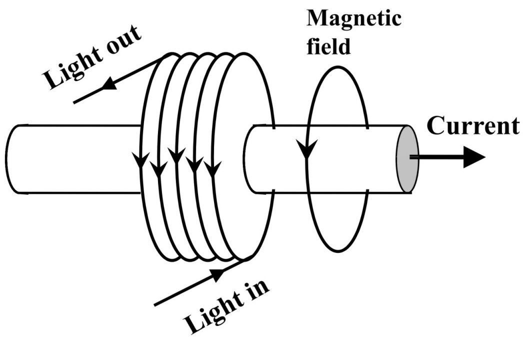


Figure 5: Measurement of high current by Faraday rotation measurement.

polarized light is allowed to propagate through the spiral optical fiber. The current produces circular magnetic field lines about the conductor. Hence, the magnetic field is along the direction of light wave propagation. So, due to Faraday rotation, polarization plane of the emergent light will be rotated with respect to incident light. By measuring the rotation angle one can measure the current.

Suppose, $I$ be the current through the conductor. If $H$ is associated magnetic field. Then, for $N$ number of turns of optical fiber we can write down the Ampere's circuital law as,

$$\oint_{N \text{ turns}} \boldsymbol{H} \cdot d\boldsymbol{l} = NI$$

or,

$$Hl = NI$$

where, $l$ is the total length of optical fiber winding of $N$ turns.

Therefore, using expression (1),

$$\theta = VHl = VNI$$

or,

$$I = \frac{\theta}{VN}$$

Here, one can measure the high current using this expression.