S_{12}, $\left(\quad \right)$

$= (1\ 3\ 8\ 10)(2)(4\ 7\ 5\ 6\ 9)$

$\alpha = (1\ 3\ 7)$, $\beta = (4\ 6\ 8)$

Disjoint cycle.

$$\begin{cases} \alpha\beta = (1\ 3\ 7)(4\ 6\ 8) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 2 & 7 & 6 & 5 & 8 & 1 & 4 \end{pmatrix} \\ \beta\alpha = (4\ 6\ 8)(1\ 2\ 7) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 2 & 7 & 6 & 5 & 8 & 1 & 4 \end{pmatrix} \end{cases}$$

$x \in \alpha$, $\alpha\beta(x) = \alpha(\beta(x)) = \alpha(x) = y$, $y \in \alpha$.

$\beta\alpha(x) = \beta(\alpha(x)) = \beta(y) = y$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 4 & 1 & 5 & 2 \end{pmatrix}$$

$= (1\ 3\ 4)(2\ 6) = (2\ 6)(1\ 3\ 4)$

Product of two disjoint cycles commutes.

$\varepsilon = (1)$

Let $\alpha = (1\ 3\ 7)(3\ 2\ 6)$

$= (3\ 2\ 6\ 7\ 1)$

$\beta = (1\ 2\ 4\ 6)(2\ 5\ 6\ 7)$

$= (2\ 5\ 1)(4\ 6\ 7)$

Thm:— Every permutation of a finite set can be written as a cycle or a product of disjoint cycles.

$\Rightarrow$ $A = \{1, 2, \ldots, n\}$. consider S_n.

$\alpha \in S$

... S_n.

Let $a_1 \in A$, $\alpha(a_1) = a_2$,

$\alpha(a_2) = a_3 = \alpha(\alpha(a_1)) = \alpha^2(a_1)$

we proceed until we get $a_1 = \alpha^m(a_1)$ for some m.

$\alpha^i(a_1) = \alpha^j(a_1)$, $i < j$

$\Rightarrow a_1 = \alpha^{j-i}(a_1) \longrightarrow$ first repetation.

$j - i = m$

$a_1, \alpha(a_1) = a_2, \alpha(a_2) = a_3 = \alpha^2(a_1)$

$\ldots \alpha^m(a_1) = a_1$

$\alpha^{m-1}(a_1) = a_m$, $\alpha(a_m) = a_1$

we have a cycle $(a_1\, a_2\, a_3 \ldots a_m)$

$\alpha = (a_1\, a_2\, a_3 \ldots a_m) \ldots$

$b_1 \in A$, $b_1 \neq a_i$, $i = 1(1)m$.

$b_2 = \alpha(b_1)$, $b_3 = \alpha(b_2) \ldots$, $b_1 = \alpha(b_k)$

we have another cycle $(b_1\, b_2 \ldots b_k)$

....

$\alpha = (a_1\, a_2 \ldots a_m)(b_1\, b_2 \ldots b_k)$

$\ldots (d_1\, d_2 \ldots d_\ell)$

Prop: — Product of disjoint cycles commute.

$\Rightarrow$ S_n. Let $\alpha = (a_1\, a_2 \ldots a_m)$

$B = (b_1\, b_2\, \ldots\, b_k)$

$\beta = (b_1, b_2, \ldots, b_k)$

claim, $\alpha\beta = \beta\alpha$.

Let $A = \{a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_k, c_1, c_2, \ldots, c_\ell\}$

claim:- $\alpha\beta(x) = \beta\alpha(x)$, $\forall x \in A$. $\quad m + k + \ell = n$.

Let $x \in \alpha$ i.e. $x = a_i$, $i = 1(1)m$.

$\alpha\beta(x) = \alpha\beta(a_i) = \alpha(a_i) = a_{i+1}$

[if $i = m$, then $a_{i+1} = a_1$]

$\beta\alpha(x) = \beta\alpha(a_i) = \beta(\alpha(a_i))$

$= \beta(a_{i+1}) = a_{i+1}$

$\therefore \alpha\beta(x) = \beta\alpha(x)$, $\forall x \in \alpha$.

Similarly, $\alpha\beta(x) = \beta\alpha(x)$, $\forall x \in \beta$.

$x \in \{c_1, c_2, \ldots, c_\ell\}$.

α, β both fix c_j's. Let $x = c_j$.

$\alpha\beta(x) = \alpha\beta(c_j) = \alpha(c_j) = c_j$

$\beta\alpha(x) = \beta\alpha(c_j) = \beta(c_j) = c_j$

$\therefore \alpha\beta(x) = \beta\alpha(x)$, $\forall x \in A$.

Order of a permutation:-

$\alpha = (a_1\ a_2\ \dots\ a_k)$ k-cycle.

Then order of $\alpha = k$ as $\alpha^k(a_i) = a_i$

In S_3, $\rho_1 = (1\ 2\ 3)$, 3-cycle.

$|\rho_1| = 3$ $\quad \rho_1^3 = \rho_0$.

$\rho_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $\rho_1^2 = \rho_2$, $\rho_1^3 = \rho_0$

Let $\beta = (a_1\ a_2\ \dots\ a_m)(b_1\ b_2\ \dots\ b_k)$

product of disjoint cycles

$\beta = \beta_1 \cdot \beta_2$

$|\beta_1| = m$, $|\beta_2| = k$.

$$|\beta| = \text{l.c.m.}(|\beta_1|, |\beta_2|)$$
$$= \text{l.c.m.}(m, k)$$

Problem :— In S_7, what is the highest order of a permutation.

$\Rightarrow$ $\alpha \to$ 1-cycle, 2-cycle ..., 7cycle

$(1\ 3)$ $\quad (1\ 3\ 4\ 7\ 2\ 6\ 5)$

product $\to$ $\underbrace{(2)(5)}_{10}$ $\quad \underbrace{(3)(4)}_{12}$ $\quad \underbrace{(2)(2)(3)}_{6}$

$\alpha = \alpha_1 \cdot \alpha_2$, $\quad \alpha_1$ 3 cycle
$\quad \alpha_2$ 4 cycle.

$\alpha = (1\ 3\ 7)(4\ 5\ 2\ 6)$

$|\alpha| = 12$ highest in S_7.

In S_{10},

$$2 \quad 3 \quad 5 \;\to\; 30 \qquad\qquad 3 \quad 7 \;\to\; 21$$

$$\alpha = (1\ 2)(3\ 4\ 5)(6\ 7\ 8\ 9\ 10)$$

$|\alpha| = 30$.

2-cycle / Transposition :—

$(1\ 2)$.

Any permutation can be written as a product of 2-cycles.

$$(1\ 2\ 3\ 4\ 5)$$
$$= (1\ 5)(1\ 4)(1\ 3)(1\ 2)$$

$$(1\ 2\ 3\ 4\ 5)$$

$1 \to 2$
$2 \to 3$
$3 \to 4$
$4 \to 5$
$5 \to 1$

$$(7\ 4\ 3\ 1\ 5\ 6\ 2) = (5\ 6\ 2\ 7\ 4\ 3\ 1)$$
$$= (7\ 2)(7\ 6)(7\ 5)(7\ 1)(7\ 3)(7\ 4)$$
$$= (5\ 1)(5\ 3)(5\ 4)(5\ 7)(5\ 2)(5\ 6)$$
$$= (3\ 4)(3\ 7)(3\ 2)(3\ 6)(3\ 5)(3\ 1)$$

$$= (3\ 1\ 5\ 6\ 2\ 7\ 4)$$
$$= (7\ 4\ 3\ 1\ 5\ 6\ 2)$$

$\alpha = (1\ 2\ 3\ 4\ 5)$ $\quad$ α is even

$= (5\ 1)(1\ 4)(3\ 1)(2\ 1)$

$(14) = (41)$, $(3\ 1) = (1\ 3)$

$\alpha = (1\ 3\ 4\ 6)(2\ 7\ 10)(5\ \ 9)$

$= (16)(14)(13)(2\ 10)(27)(5\ 9)$

6 2-cycles. α is even

$\beta = (1\ 2\ 3\ 4)$

$= (1\ 4)(1\ 3)(1\ 2)$ odd

$\varepsilon = (1)$ even $\quad$ 0 2-cycle

S_3, ρ_0, ρ_1, ρ_2 even
(1) $(1\ 2\ 3)$ $(2\ 3\ 1)$

ρ_3, ρ_4, ρ_5 odd
$(2\ 3)$ $(1\ 3)$ $(1\ 2)$

$S_n \longrightarrow$ No of even permutation $= \frac{n!}{2}$

" " odd " $= \frac{n!}{2}$.