Permutation Group

$A = \{a, b, c\}$

abc   acb
bca   cba
cab   bac

3!

a, b ab   ba   2!

$a, b, c, d \longrightarrow 4!$

$a \xrightarrow{f} c$
$b \longrightarrow b$
$c \longrightarrow a$

$f : A \to A$, bijective mapping
f is called a permutation of A.

$A = \{1, 2, 3\}$

$g : A \to A$,
$$\begin{cases} g(1) = 2 \\ g(2) = 3 \\ g(3) = 1 \end{cases}$$

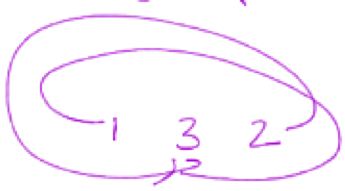
g is a permutation on A.

$$g = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$f_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \qquad f_3 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$f_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \qquad f_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \qquad f_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$



$G = \{f_0, f_1, f_2, f_3, f_4, f_5\}$

$$f_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$f : A \to B \quad , \quad g : B \to C$

$g \circ f : A \to C$

$$f_1 . f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = f_0$$

$$\begin{cases} f_1 . f_2 (1) = f_1(f_2(1)) = f_1(3) = 1 \\ f_1 . f_2 (2) = f_1(f_2(2)) = f_1(1) = 2 \\ f_1 . f_2 (3) = f_1(f_2(3)) = f_1(2) = 3 \end{cases}$$

$$f_2 f_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = f_3$$

$$f_4 . f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = f_5$$

$$f_4 f_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = f_1$$

<u>Cayley table</u> : $\longrightarrow$

	f_0	f_1	f_2	f_3	f_4	f_5
f_0						
f_1						
f_2						
f_3						
f_4						
f_5						

ρ_0 is the id element.

$$\rho_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad \rho_1^{-1} = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \rho_2.$$

$$\rho_1 \cdot \rho_2 = \rho_0 = \rho_2 \cdot \rho_1$$

$$\rho_1^{-1} = \rho_2, \quad \rho_2^{-1} = \rho_1,$$

$$\rho_3^{-1} = \rho_3, \quad \rho_4^{-1} = \rho_4, \quad \rho_5^{-1} = \rho_5$$

$G = \{\rho_0, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5\}$ is a permutation group of three elements with order 6.

Symmetric group S_3 of order 6.

$A = \{1, 2, 3, \ldots, n\}$

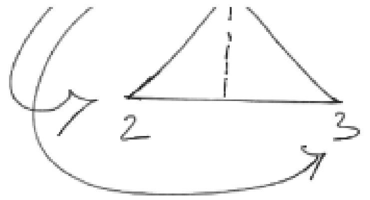
S_n is the permutation group of order $n!$ Symmetric group of degree n.

---

is $S_3 \cong D_3$??

$$\rho_0 = R_0 \qquad\qquad \rho_3 = f_1$$
$$\rho_1 = R_{120} \qquad\qquad \rho_4 = f_2$$
$$\rho_2 = R_{240} \qquad\qquad \rho_5 = f_3$$

$0 - (1 \quad 2 \quad 3) - \rho$



$R_0 = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} = \rho_0$

$R_{120} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \rho_1$

$R_{240} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \rho_2$

$f_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \rho_3$

$f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \rho_4$

$f_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \rho_5$

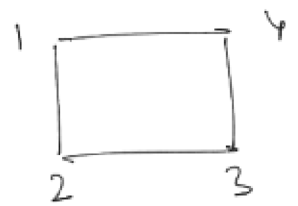
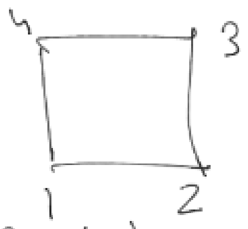
We know, $R_{120}^3 = R_0 = R_{240}^3$

$$\rho_1^3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = \rho_0$$

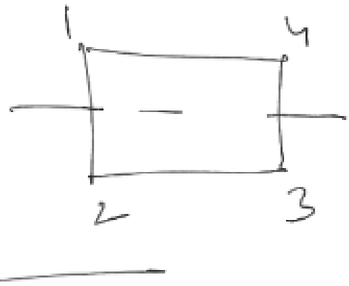
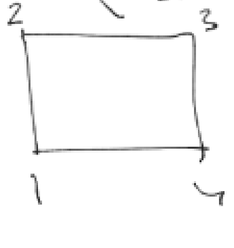
$\rho_2^3 = \rho_0$

S_3 is Non-abelian. $\therefore S_3 \cong D_3$.

$D_n \leq S_n$

$D_4 \rightarrow$  $\longrightarrow R_{90}$ 

$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$

 $\xrightarrow{H}$  $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$

Non-abelian group upto order 8 :-

order 6 $\longrightarrow$ D_3 or S_3

order 8 $\longrightarrow$ D_4 and $Q(8)$

---

order 1 $\longrightarrow$ $\{0\}$ upto isomorphism

2 $\longrightarrow$ $\mathbb{Z}_2$

3 $\longrightarrow$ $\mathbb{Z}_3$

4 $\longrightarrow$ $\mathbb{Z}_4$ and K-4 group

5 $\longrightarrow$ $\mathbb{Z}_5$

6 $\longrightarrow$ $\mathbb{Z}_6$, S_3 (non abelian)

7 $\longrightarrow$ $\mathbb{Z}_7$

8 $\longrightarrow$ $\mathbb{Z}_8$, $\mathbb{Z}_4 \oplus \mathbb{Z}_2$, $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$,

D_4, Q_8 (non abelian)

---

cycle notation :—

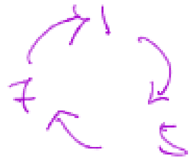
S_{10}, $\quad \alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 3 & 4 & 1 & 5 & 10 & 9 & 8 & 7 & 6 \end{pmatrix}$

$= (1\ 2\ 3\ 4)(5)(6\ 10)(7\ 9)(8)$

$= (1\ 2\ 3\ 4)(6\ 10)(7\ 9)$

S_9 $\quad (1\ 5\ 7)(2\ 4\ 8)$

$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 4 & 3 & 8 & 7 & 6 & 1 & 2 & 9 \end{pmatrix}$



S_8, $\quad \alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 4 & 1 & 6 & 3 & 2 & 7 & 8 \end{pmatrix}$

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 5 & 8 & 4 & 6 & 2 & 1 & 3 \end{pmatrix}$$

$$\alpha = (1\ 5\ 3)(2\ 4\ 6)$$

$$\beta = (1\ 7)(2\ 5\ 6)(3\ 8)$$

$$\alpha \cdot \beta = (1\ 5\ 3)(2\ 4\ 6)(1\ 7)(2\ 5\ 6)(3\ 8)$$

$$= (1\ 7\ 5\ 2\ 3\ 8)(4\ 6)$$

$$\begin{cases} \alpha\cdot\beta(1) = 7 \\ \alpha\cdot\beta(7) = 5 \\ \alpha\cdot\beta(5) = 2 \\ \alpha\cdot\beta(2) = 3 \\ \alpha\cdot\beta(3) = 8 \\ \alpha\cdot\beta(8) = 1 \\ \alpha\cdot\beta(4) = 6 \\ \alpha\cdot\beta(6) = \end{cases} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 3 & 8 & 6 & 2 & 4 & 5 & 1 \end{pmatrix}$$

$$\beta \cdot \alpha = (1\ 7)(2\ 5\ 6)(3\ 8)(1\ 5\ 3)(2\ 4\ 6)$$

$$= (1\ 6\ 5\ 8\ 3\ 7)(2\ 4)$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 4 & 7 & 2 & 8 & 5 & 1 & 3 \end{pmatrix}$$