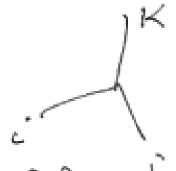
The Quaternions :—

(a,b) $\quad (a + bi)(c + di) = (ac - bd) + i(ad + bc)$

(a,b,c) $\quad (a_1 + b_1 i + c_1 j)\,(a_2 + b_2 i + c_2 j)$

$i \cdot j = k$



$(\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, +)$ group under addition by componentwise.

Let $1 = (1, 0, 0, 0)$, $i = (0, 1, 0, 0)$,

$j = (0, 0, 1, 0)$, $k = (0, 0, 0, 1)$.

$a \equiv (a_1, a_2, a_3, a_4) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$

$(a_1, a_2, a_3, a_4) = a_1 \cdot 1 + a_2 i + a_3 j + a_4 k.$

Addition :—

$$(a_1 + a_2 i + a_3 j + a_4 k) + (b_1 + b_2 i + b_3 j + b_4 k)$$
$$= (a_1 + b_1) + (a_2 + b_2) i + (a_3 + b_3) j + (a_4 + b_4) k.$$

Multiplication :—

$1 \cdot a = a \cdot 1 = a$, $\quad i^2 = j^2 = k^2 = -1$

$ij = k$, $jk = i$, $ki = j$

$ji = -k$, $kj = -i$, $ik = -j$

$$(a_1 + a_2 i + a_3 j + a_4 k) \cdot (b_1 + b_2 i + b_3 j + b_4 k)$$
$$= (a_1 b_1 - a_2 b_2 - a_3 b_3 - a_4 b_4)$$

$(i\, j\, k)$ +
$(i\, k\, j)$ −

$$+ i(a_1 b_2 + a_2 b_1 + a_3 b_4 - a_4 b_3)$$
$$+ j(a_1 b_3 + a_3 b_1 + a_4 b_2 - a_2 b_4)$$
$$+ k(a_1 b_4 + a_4 b_1 + a_2 b_3 - a_3 b_2)$$

$ij = k \quad , \quad ji = -k.$

$\therefore$ multiplication is not commutative.

1 is the id element.

Inv :— $(a_1 + a_2 i + a_3 j + a_4 k)(a_1 - a_2 i - a_3 j - a_4 k)$

$$= a_1^2 + a_2^2 + a_3^2 + a_4^2$$

$$|a|^2 = a_1^2 + a_2^2 + a_3^2 + a_4^2$$

$$\bar{a} = a_1 - a_2 i - a_3 j - a_4 k$$

$$a . \bar{a} = |a|^2$$

$$\Rightarrow \quad a . \frac{\bar{a}}{|a|^2} = 1$$

$(\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \cdot)$ non abelian group.

$(\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, +, \cdot)$ is a skew field.

+	·	
Clo	clo	$\Rightarrow$ Ring
Asso	Asso	
Id	Id	
Inv	Inv (nonzero)	
Com	Com	

Distributive

$a.(b+c) = a.b + a.c \quad , \forall a, b, c$

Field – Comm. w.r.t. to multiplication field
$\Rightarrow$ Skew field.

Let $G = \{\pm 1, \pm i, \pm j, \pm k\}$ is a group of order 8 under quaternion multiplication.

Here $i^4 = 1$, $j^2 = i^2$, $ji = i^3 j$

$$ji = -k = -ij = (-i)j = i^3 j$$

$$G = \{e, a, a^2, a^3, b, ab, a^2 b, a^3 b\}$$
$$= \{1, i, i^2, i^3, j, ij, i^2 j, i^3 j\}$$
$$= \{1, i, -1, -i, j, k, -j, -k\}$$

$$G = \{a, b : a^4 = e;\ b^2 = a^2;\ ba = a^3 b\}$$

$$D_4 \not\cong G.$$

$b^2 = 1$ $\quad\quad$ $b^2 = a^2 = -1$

Another representation :—

Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$.

$B^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = A^2$ $\quad$ $AB = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

$B^3 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -B$ $\quad$ $AB^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$B^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$ $\quad$ $AB^3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$

$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = B^2$

$$A^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -A$$

$$A^4 = \hat{I}.$$

$$A^2 = B^2, \quad B^4 = \hat{I}, \quad AB = B^3 A.$$

$$G = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \right\}$$

is a quaternion group of order 8 under matrix multiplication (Matrix form)

$$Q(8) = \{ a, b : a^4 = 1,\ b^2 = a^2;\ ba = a^3 b \}.$$

quaternion group of order 8.

Determine all groups of order 8 upto isomorphism :—

Three abelian groups :—

$$\mathbb{Z}_8,\ \mathbb{Z}_4 \oplus \mathbb{Z}_2,\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

Two non-abelian groups :—

$$D_4,\ Q_8\ (Q(8))$$

D_4	Q_8
$(a^2 b)(a^3 b)$	$(a^2 b)(a^3 b)$

$= (a^2 b)(b a)$

$= a^2 b^2 a$

$= a^3$

$= (a^2 b)(b a)$

$= a^2 b^2 a$

$= a^2 a^2 a = a^5 = a$

1) find the cayley table of $Q(8)$.

a) find the center $Z(Q(8))$

b) find $C(a)$

c) find $C(b)$

d) find all cyclic subgroup.

H/W