Problem :—

1) Let G has exactly one proper nontrivial subgroup. Then G is cyclic.

$$\{0\} < \overset{H}{\langle 2 \rangle} < \mathbb{Z}_4$$

$$\{0\} < \langle 3 \rangle < \mathbb{Z}_9$$

$$\underline{\langle 2 \rangle, \langle 4 \rangle, \langle 8 \rangle} \; \mathbb{Z}_{16}$$

(order of $G = p^2$)
$\{e\} < H < G$
$|H| = p$)

$$\{0\} < \langle p \rangle < \mathbb{Z}_{p^2}$$

$\Rightarrow$

Let H is the only proper nontrivial subgroup of G.

$H \neq \{e\}$, $H < G \Rightarrow G \setminus H \neq \phi$.

Let $a \in G \setminus H$ $(a \neq e)$

$\langle a \rangle$ is a subgroup of G.

$\langle a \rangle = G$ or H.

If $\langle a \rangle = H \Rightarrow a \in H$, contradiction.

$\therefore \langle a \rangle = G$. G is cyclic.

$\mathbb{Z}_4$. $\langle 2 \rangle < \mathbb{Z}_4$

$3 \in \mathbb{Z}_4 \setminus \langle 2 \rangle$, $\mathbb{Z}_4 = \langle 3 \rangle$

$\mathbb{Z}_9$, $\langle 3 \rangle < \mathbb{Z}_9$

$1, 2, 4, 5, 7, 8 \in \mathbb{Z}_9 \setminus \langle 3 \rangle$

$\mathbb{Z}_9 = \langle 1 \rangle, \langle 2 \rangle, \langle 4 \rangle, \langle 5 \rangle, \langle 7 \rangle, \langle 8 \rangle$.

Let $|G| = p^2$. If G has exactly one proper nontrivial subgroup H.

Then $|H| = ?$ $\qquad |H| = p.$

2) Let G has exactly two proper nontrivial subgroups? What can you say about order of G?

$\Rightarrow$ $|G| = p \cdot q$

Let H and K be two proper subgroups of G.

$H = \langle a \rangle$ of order p

$K = \langle b \rangle$ of order q.

$|\mathbb{Z}_6| = 2.3 \qquad \{0\} < \overset{3}{\langle 2 \rangle}, \overset{2}{\langle 3 \rangle} < \mathbb{Z}_6$

$|\mathbb{Z}_{10}| = 2.5 \qquad \{0\} < \overset{5}{\langle 2 \rangle}, \overset{2}{\langle 5 \rangle} < \mathbb{Z}_{10}$

$|\mathbb{Z}_{15}| = 3.5 \qquad \{0\} < \langle 3 \rangle, \langle 5 \rangle < \mathbb{Z}_{15}$

$\mathbb{Z}_{pq} \qquad \{0\} < \langle p \rangle, \langle q \rangle < \mathbb{Z}_{pq}$

Converse is not true :—

$|D_3| = 6 = 2.3$

$\{R_0, R_{120}, R_{240}\}, \{R_0, f_1\}, \{R_0, f_2\}$
$\{R_0, f_3\}$.

Theorem :—

Every group of prime order is cyclic.

$\Rightarrow$ Let $|G| = p$.

Let $a\ (\neq e) \in G$.

$\langle a \rangle$ is a nontrivial subgroup of G.

$|\langle a \rangle| \,/\, |G|$ [order of subgp divides order of group]

$\Rightarrow |\langle a \rangle| \,/\, p$

$\Rightarrow |\langle a \rangle| = 1$ or p

$\Rightarrow |\langle a \rangle| = p$ [$\because \langle a \rangle \neq \{e\}$]

$\Rightarrow G = \langle a \rangle$.

$$|Z_{p^2 q}| = p^2 q$$

order

$$\begin{tikzcd}
 & p^2 q \arrow[dl, dash] \arrow[dr, dash] & \\
pq \arrow[d, dash] \arrow[drr, dash] & & p^2 \arrow[d, dash] \\
q \arrow[dr, dash] & & p \arrow[dl, dash] \\
 & 1 &
\end{tikzcd}$$

Subgroup lattice

$$\begin{tikzcd}
 & \langle 1 \rangle \arrow[dl, dash] \arrow[dr, dash] & \\
\langle p \rangle \arrow[d, dash] & & \langle q \rangle \arrow[d, dash] \\
\langle p^2 \rangle \arrow[dr, dash] & & \langle pq \rangle \arrow[dl, dash] \\
 & \langle 0 \rangle &
\end{tikzcd}$$

D_p

$|D_{13}| = 26$

$|D| = 30$

$|15| = 30$

6   R 15   2

10   cyclic

1, 3, 5, 15

D_{15} $\quad \langle R_0, R_{\frac{360}{15}}, \ldots, R_{14 \cdot \frac{360}{15}} \rangle$

cyclic

$\langle R_0, R_{60}, R_{120}, R_{180}, R_{240}, R_{300}, R_{360} \rangle$

D_n, n is odd but not prime.

D_{15}

D_{15}

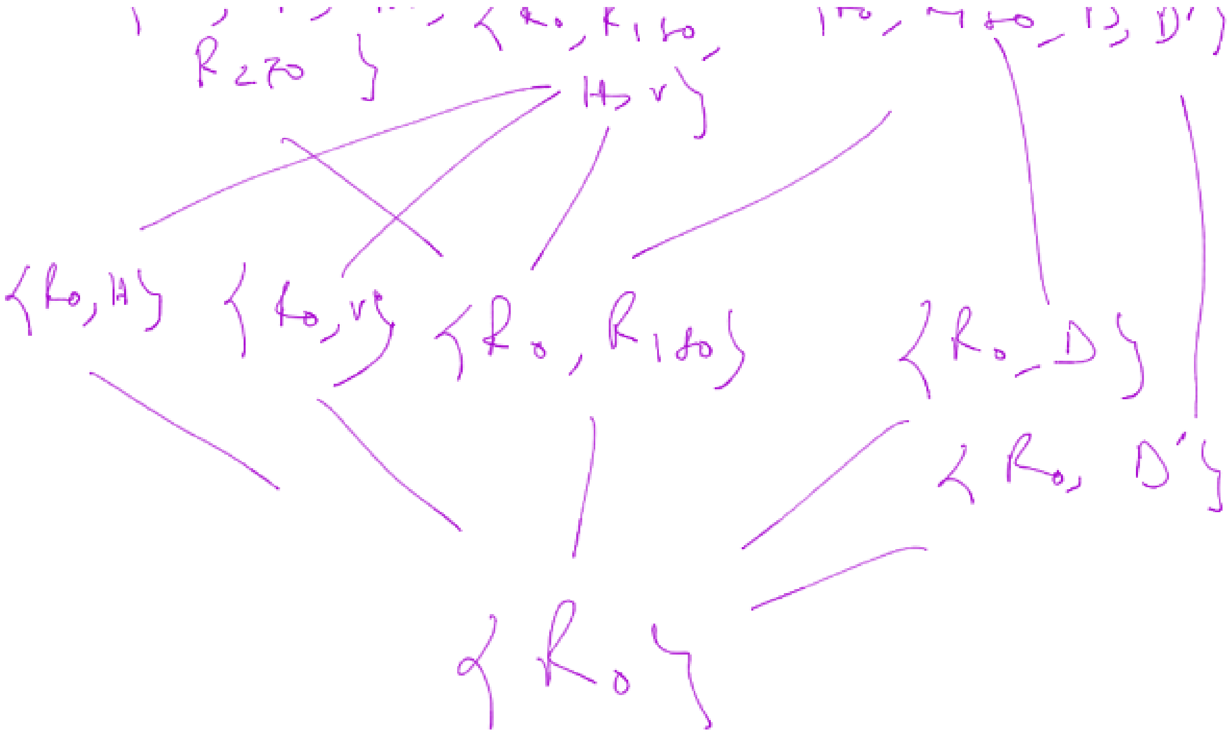
R_{15} $\{R_0, f_1\}, \ldots, \{R_0, f_{15}\}$

$\langle R_0, R_{120}, R_{240} \rangle$ $\langle R_0, R_{72}, R_{144}, R_{216}, R_{288} \rangle$

$\langle R_0 \rangle$

D_4

$\langle R_0, R_{90}, R_{180}$ ...


R270 }
{R0, R180, H, V}
R180, D, D'}
{R0, H}
{R0, V}
{R0, R180}
{R0, D}
{R0, D'}
{R0}