Group :—

Group
- i) closure prop. (Groupoid)
- ii) Associative prop $\longrightarrow$ (Semigroup)
- iii) Existence of identity $\longrightarrow$ (Monoid)
- iv) Existence of inverse
- v) Commutative prop $\longrightarrow$ Abelian group.

Ex :— 1) $(\mathbb{Z}, \cdot)$ is monoid.

$(\mathbb{R}, \cdot)$ $\quad a$ $\quad |-5.1| = |-5| = 5$

$a * b = |ab|$ $\quad |-a.b| = -a = a$

Semigroup.

① Uniqueness of identity :—

In a group G, $\exists$ only one identity element.

$\Rightarrow$ Let G has two identity element e and e_1 (say).

Then $a.e = a = e.a$, $\forall a \in G \longrightarrow (1)$

$a.e_1 = a = e_1.a$, $\forall a \in G. \longrightarrow (2)$

from (1), $e_1 = e.e_1$

from (2), $e.e_1 = e$

$\Rightarrow e_1 = e$.

② Cancellation law :—

In a group G, for $a, b, c \in G$, we have, ① $a.b = a.c \Rightarrow b = c$ [Left cancellation]

② $b.a = c.a \Rightarrow b = c$ [Right cancellation]

$\Rightarrow$ Let $a.b = a.c$

Premultiplying both side by a^{-1},

$$a^{-1}.(a.b) = a^{-1}.(a.c)$$

$\Rightarrow (a^{-1}.a).b = (a^{-1}.a).c$ [by associative law]

$\Rightarrow e.b = e.c$ [e is the id element of G]

$\Rightarrow b = c$

3) Uniqueness of inverse :—

For each $a \in G$, $\exists$ a unique element $b \in G$ s.t. $a.b = b.a = e$.

$\Rightarrow$ Let b and c be two inverses of a.

Then $\left.\begin{array}{l} a.b = e = b.a \\ a.c = e = c.a \end{array}\right\}$

$$\left\{\begin{array}{l} \cancel{a.b = a.c} \\ \cancel{\Rightarrow b = c} \end{array}\right.$$ [~~by~~ left cancellation]

$$b = b.e = b.(a.c) = (b.a).c$$

$$= e.c = c$$

4) $(a^{-1})^{-1} = a, \forall a \in G$.

$a^{-1}.a = e$ —① $\rightarrow$ a is the inverse of a^{-1} i.e. $a = (a^{-1})^{-1}$.

$a^{-1}.(a^{-1})^{-1} = e$ —②

$\therefore a^{-1}.a = a^{-1}.(a^{-1})^{-1}$

$\Rightarrow a = (a^{-1})^{-1}$ [cancellation]

5) In $(G, \circ)$, $(a \circ b)^{-1} = b^{-1} \circ a^{-1}, \forall a, b \in G$.

$$(a \circ b) \circ (b^{-1} \circ a^{-1}) = a \circ (b \circ b^{-1}) \circ a^{-1}$$

$= a \circ e \circ a^{-1}$

$= a \circ a^{-1} = e.$

$\therefore (a \circ b)^{-1} = b^{-1} \circ a^{-1}.$

6) $(G, *)$ is a group. The eq^n $a * x = b$ and $y * a = b$ have unique solution in G for x, y.

$\Rightarrow$ $a * x = b$

Premultiplying both side by a^{-1},

$a^{-1} * (a * x) = a^{-1} * b$

$\Rightarrow (a^{-1} * a) * x = a^{-1} * b$ [Asso]

$\Rightarrow e * x = a^{-1} * b$ [e is the id elem]

$\Rightarrow x = a^{-1} * b.$

So $a * x = b$ has a solution $a^{-1} * b$.

Uniqueness :— Let x_1, x_2 be two solutions of $a * x = b$

$a * x_1 = b = a * x_2$

$\Rightarrow x_1 = x_2$ [left cancellation]

$y * a = b \longrightarrow$ H/W

7) A semigroup $(S, *)$ is a group iff $\forall a, b \in S$, the equations $a * x = b$ and $y * a = b$ have solutions in S for x & y.

$\Rightarrow$ Let $(S, *)$ be a semigroup and $a * x = b$ and $y * a = b$ have solutions in S for x, y, $\forall a, b \in S$.

Let $a * e = a$, $e' * a = a$ for some $e, e' \in S$.

for $c \in S$, $a * x_1 = c$, $y_1 * a = c$, for some $x_1, y_1 \in S$.

Now, $c * e = (y_1 * a) * e$

$= y_1 * (a * e)$ [Asso]

$= y_1 * a = c$

Since c is arbitrary, so $a * e = a$, —① $\forall a \in S$

$e' * c = e' * (a * x_1)$

$= (e' * a) * x_1 = a * x_1 = c$.

Since c is arbitrary, so $e' * a = a$, —② $\forall a \in S$.

$e' * e = e'$ [from ①]

$e' * e = e$ [from ②]

$\Rightarrow e = e'$.

$a * e = e * a = a$, $\forall a \in S$.

$\therefore$ e is the id. element of S.

Existence of inv :—

$a * x_2 = e$, $y_2 * a = e$, for some $x_2, y_2 \in S$.

$(y_2 * a) * x_2 = y_2 * (a * x_2)$

$\Rightarrow e * x_2 = y_2 * e$

$\Rightarrow x_2 = y_2 = b$ (say).

Then $a * b = e = b * a$

$\therefore$ b is the inverse of a.

Since a, b arbitrary, so inverse prop. holds. $\therefore$ $(S, *)$ is a group.

Converse part :— By ...

[illegible] by prev. prop.

H/W

Quasi group :—

A groupoid $(G, \circ)$ is said to be a quasigroup if for any two elements $a, b \in G$, each of the eq^n $a \circ x = b$ and $y \circ a = b$ has a unique solution.

Ex:— $(\mathbb{Z}, +)$ is a quasigroup.

$(\mathbb{Z}, \cdot)$ is not a " "

$2, 3 \in \mathbb{Z}$ but $2 \cdot x = 3$ has no sol in $\mathbb{Z}$.

$(\mathbb{Z}, -)$ is a quasi group but not a semigroup.

$a - x = b \Rightarrow x = a - b$ unique

$y - a = b \Rightarrow y = a + b$ "