$(\mathbb{Z}_7^*, \cdot)$

$$\bar{2} \cdot \bar{3} = \bar{6}$$

$$\bar{9} \cdot \bar{3} = \overline{27} = \bar{6}$$

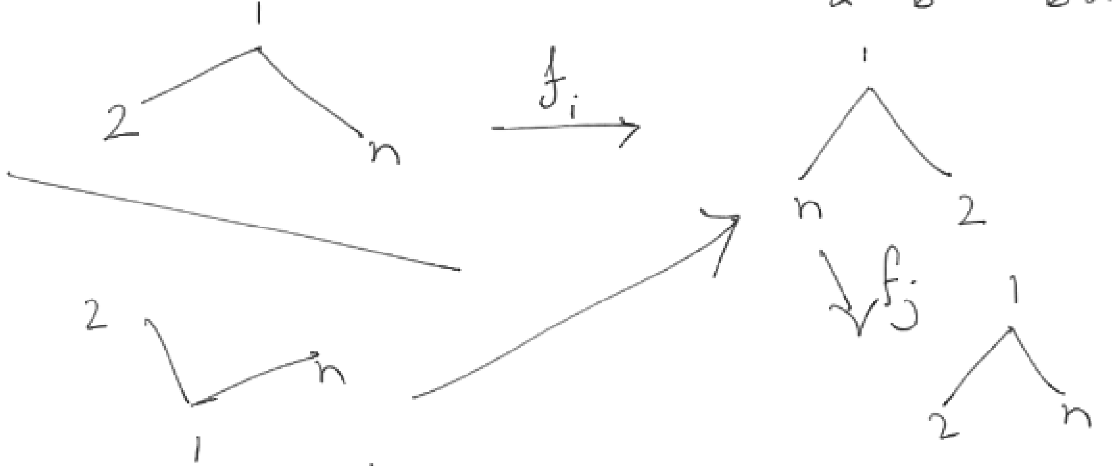
$$\bar{a} \cdot \bar{b} = \overline{ab} \equiv r \pmod{7}$$
$$= \bar{r}.$$

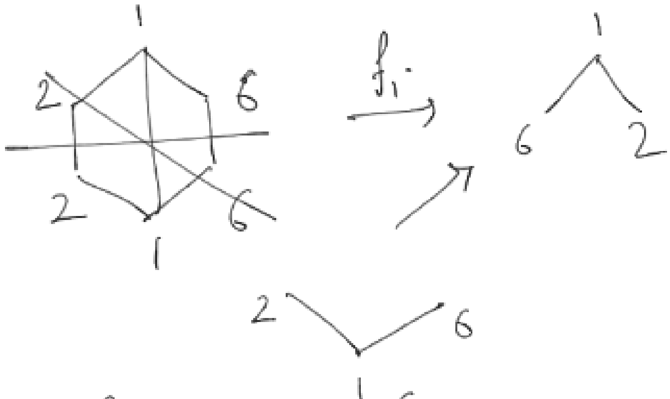
$(\mathbb{N}, *)$ $\quad a * b = l.c.m.(a, b)$

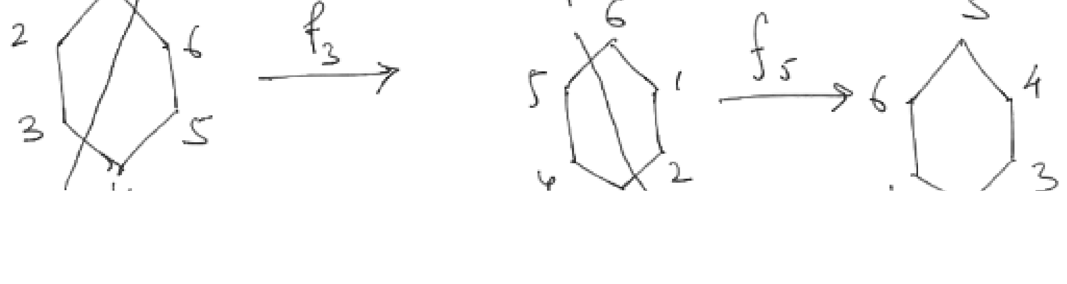
$$a * 1 = l.c.m.(a, 1) = a.$$

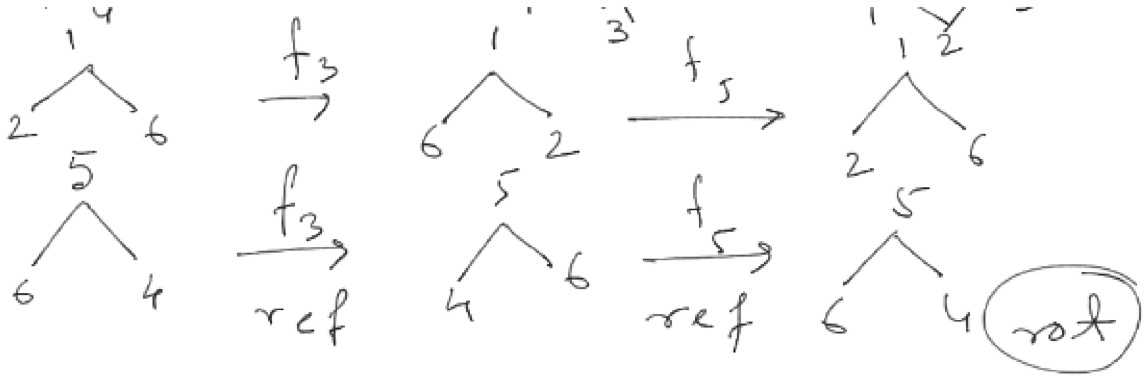
$$a * b = 1$$
$$\Rightarrow l.c.m.(a, b) = 1$$
$$\Rightarrow a = b = 1.$$

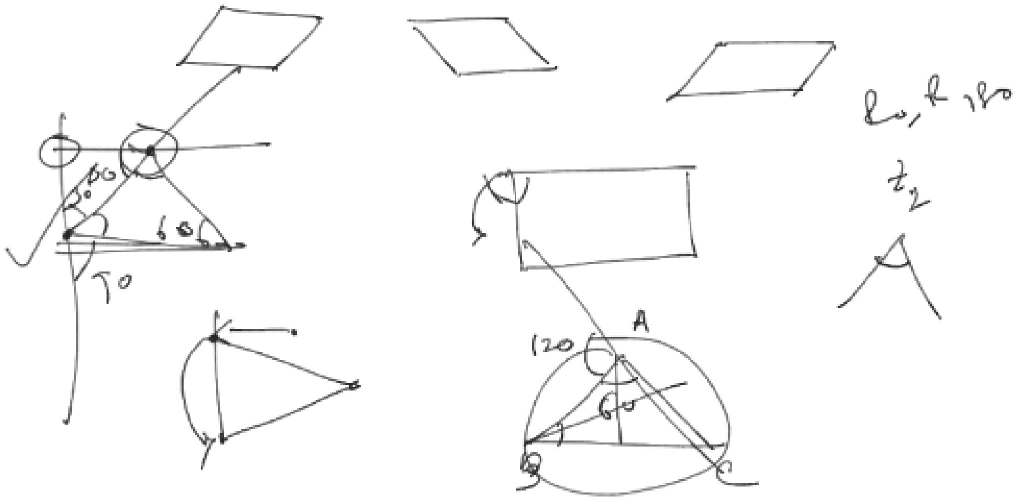
$$D_n = \{a, b : a^n = e = b^2;\ a^r b = b a^{n-r}\}$$











$a = a^{-1}, \ \forall a \in G.$

$a, b \in G.$

<u>claim</u> $ab = ba$

$(ab)^{-1} = b^{-1} a^{-1}$

$\Rightarrow ab = ba$

<u>Group</u>

<u>Examples</u>:—

1) $G \subseteq \mathbb{C}$, $G = \{1, -1, i, -i\}$

$(G, \cdot)$ $\quad \cdot \equiv$ complex multiplication.

$-1 \cdot -1 = 1$, $i(-i) = 1$, $(-i)\,i = 1$.

$i^2 = -1$, $i^3 = -i$, $i^4 = 1$

$(-i)^4 = 1$.

$$G = \{R_0, R_{90}, R_{180}, R_{270}\}$$

$$\begin{array}{cccc} \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 4 & 2 & 4 \end{array}$$

(Isomorphic)

$$\{1, i, -1, -i\}$$

$$U(10) = \{1, 3, 7, 9\}$$

$$\begin{array}{cccc} \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 4 & 4 & 2 \end{array}$$

$3^2 = 9 \equiv -1 \pmod{10}$

$3^4 \equiv 1 \pmod{10}$

$$Z_4 = \{0, 1, 2, 3\}$$

$$\begin{array}{cccc} \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 4 & 2 & 4 \end{array}$$

$3+3 = 6 = 2$, $3+3+3 = 1$, $3+3+3+3 = 0$

G, Z_4, $U(10)$, The set of all rotations in D_4.

(Isomorphic)

2) $G = \{1, \omega, \omega^2\}$ group

$Z_3 = \{0, 1, 2\}$

The set of all rotations in D_3.

$$= \{R_0, R_{120}, R_{240}\}$$

$$\begin{array}{cc} \downarrow & \downarrow \\ 3 & 3 \end{array}$$

3) $(\mathbb{R}^* - \mathbb{Q}^*, \cdot)$ is not a group.

'$\cdot$' is not closed.

4) Set of all real matrices $M(\mathbb{R})$
$(M(\mathbb{R}), +)$, $(M(\mathbb{R}), \cdot)$ is not a group.

5) $M_2(\mathbb{R}) \equiv$ The set of all 2×2 real matrices.

$(M_2(\mathbb{R}), +)$ is a group. $\underline{\underline{M_n(\mathbb{R})}}$

But $(M_2^*(\mathbb{R}), \cdot)$ is not a group.

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ has no inverse.

6) $GL(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc \neq 0; \; a, b, c, d \in \mathbb{R} \right\}$

General linear group of 2×2 matrices over $\mathbb{R}$.

$GL(2, \mathbb{R})$ is a non-abelian group.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

7) $SL(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1; \; a, b, c, d \in \mathbb{R} \right\}$

Special linear group of 2×2 real matrices. $SL(2, \mathbb{R}) \subseteq GL(2, \mathbb{R})$

$SL(2, \mathbb{R})$ Non-abelian group.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

8) $GL(n, \mathbb{R})$, $SL(n, \mathbb{R})$

9) $GL(n, \mathbb{Z})$ is not a group.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \notin GL(n, \mathbb{Z}).$$

$SL(n, \mathbb{Z})$ is a group.

10) $GL(n, \mathbb{Z}_p)$, $p \equiv$ prime
group.

$GL(2, \mathbb{Z}_7)$

$$\begin{pmatrix} \bar{a}_1 & \bar{b}_1 \\ \bar{c}_1 & \bar{d}_1 \end{pmatrix}\begin{pmatrix} \bar{a}_2 & \bar{b}_2 \\ \bar{c}_2 & \bar{d}_2 \end{pmatrix} = \begin{pmatrix} \overline{a_1 a_2 + b_1 c_2} & \\ & \end{pmatrix} \in GL(2, \mathbb{Z}_7)$$

Matrix multiplication is associative.

$\begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{1} \end{pmatrix}$ is the identity element.

$$A = \begin{pmatrix} \bar{2} & \bar{3} \\ \bar{4} & \bar{5} \end{pmatrix} \qquad |A| = \overline{10} - \overline{12} = -\bar{2} = \bar{5}$$

$$A^{-1} = \frac{1}{5}\begin{pmatrix} 5 & -3 \\ -4 & 2 \end{pmatrix} = 5^{-1}\begin{pmatrix} 5 & 4 \\ 3 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 5 \\ 2 & 6 \end{pmatrix} \longleftarrow = 3\begin{pmatrix} 5 & 4 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 15 & 12 \\ 9 & 6 \end{pmatrix}$$

$5 \cdot 3 = 1 \pmod 7$ $\longrightarrow$ w.r.t. '+'

w.r.t. '·' $5^{-1} = 3$, $-5 \equiv 2 \pmod 7$

$\bar{5} \cdot \bar{3} \equiv 1 \pmod 7$

$4^{-1} = 2$

$SL(n, \mathbb{Z}_p)$ Group.