Dihedral group $D_n \Rightarrow$ The group of symmetries of a regular n gon.

$\{R_0, R_{180}, H, V\} \longrightarrow$ Klein's 4 group.

$$D_4 = \{R_0, R_{90}, R_{180}, R_{270}, H, V, D, D'\}$$

$$= \{R_{90}, R_{90}^2, R_{90}^3, \underset{=R_0}{R_{90}^4}, H, \underset{=D'}{R_{90}H}, \underset{=V}{R_{90}^2 H}, \underset{=D}{R_{90}^3 H}\}$$

$R_{1\cdot\frac{360}{4}}$, $R_{3\cdot\frac{360}{4}}$

$$= \{\alpha, \alpha^2, \alpha^3, \alpha^4 = R_0, \beta, \alpha\beta, \alpha^2\beta, \alpha^3\beta\}$$

$$D_4 = \{\alpha, \beta : \alpha^4 = R_0 = \beta^2 ; \alpha\beta = \beta\alpha^3\} \quad \left[\begin{array}{l}\alpha = R_{90} \\ \beta = H\end{array}\right]$$

$\alpha\beta = \beta\alpha^3$??

α rot, β ref $\Rightarrow \alpha\beta$ is ref.

$\therefore (\alpha\beta)^{-1} = \alpha\beta$

$\Rightarrow \beta^{-1}\alpha^{-1} = \alpha\beta$

$\Rightarrow \beta\alpha^3 = \alpha\beta$

$$\left[\begin{array}{l}\beta^2 = R_0 \\ \Rightarrow \beta^{-1} = \beta \\ \alpha^4 = R_0 \\ \Rightarrow \alpha.\alpha^3 = R_0 \\ \Rightarrow \alpha^{-1} = \alpha^3\end{array}\right]$$

$$\boxed{\begin{array}{l} 3 \equiv -1 \pmod 4 \\ 2 \equiv -2 \\ 5 \equiv 1 \end{array}}$$

$\alpha^2\beta = \beta\alpha^2$, $\alpha^3\beta = \beta\alpha$

$\alpha^r\beta = \beta\alpha^{4-r}$ in case of D_4.

In D_4, what are the possibilities of α?

$$\alpha^4 = R_0, \quad \alpha = R_{90}, R_{270}$$

$$\alpha \neq R_{180} \text{ as } R_{180}^2 = R_0$$

If $\alpha = R_{180}$ $\qquad$ $\alpha = R_0$, $\begin{Bmatrix} \alpha, & \alpha^2, & \alpha^3, & \alpha^4 \\ R_0, & R_0, & R_0, & R_0 \end{Bmatrix}$

$$\begin{Bmatrix} \alpha, & \alpha^2, & \alpha^3, & \alpha^4 \\ R_{180}, & R_0, & R_{180}, & R_0 \end{Bmatrix} \quad R_{90}, R_{270} \text{ missing}$$

$$\left\{ \begin{array}{l} R_{270}, \quad R_{270}^2 = R_{540} = R_{180}, \\ R_{270}^3 = R_{810} = R_{90}, \quad R_{270}^4 = R_{1080} = R_0 \\ R_{270}^3 = R_{180} \cdot R_{270} = R_{450} = R_{90} \end{array} \right.$$

$$D_n = \{\alpha, \beta : \alpha^n = R_0 = \beta^2 ; \alpha^r \beta = \beta \alpha^{n-r}\}$$

$$(\alpha^r \beta)^{-1} = \beta^{-1} \alpha^{-r} = \beta \alpha^{n-r}$$

$\left\{ \begin{array}{ll} \alpha & \text{r.f} \\ \beta & \text{ref.} \end{array} \right.$ $\qquad$ $\left[\begin{array}{l} \alpha^n = R_0 = R_{360} \\ -r = 360 - r \end{array} \right]$

$$D_{10} = \{R_0, R_{36}, R_{72}, \ldots, R_{9 \cdot 36}, f_1, f_2, \ldots, f_{10}\}$$

$$= \{R_{36}, R_{36}^2, \ldots, R_{36}^9, R_0, f_7, R_{36} f_7, R_{36}^2 f_7, \ldots, R_{36}^9 f_7\}$$

$$\cancel{R_{36}^{4}} f_7 = R_{36}^9 f_7$$

$$\Rightarrow f_{[illegible]} = R^5 f_{[illegible]}$$

$f_7 = R_{36} f_7$

$$R_{36}^{-4} R_{36}^{4} f_7 = R_{36}^{-4} R_{36}^{9} f_7$$

$$\Rightarrow R_{36}^{-4+4} f_7 = R_{36}^{-4+9} f_7$$

$$\Rightarrow R_0 f_7 = R_{36}^{5} f_7$$

$$\Rightarrow R_0 f_7 f_7 = R_{36}^{5} f_7 f_7$$

$$\Rightarrow R_0 R_0 = R_{36}^{5} R_0$$

$$\Rightarrow R_0 = R_{36}^{5}$$

Conclusion :- $R_{36}^{p} f_7 \neq R_{36}^{q} f_7$

where $p \neq q$.

$$D_{10} = \{ R_{36}, f_7 : R_{36}^{10} = R_0 = f_7^{2} ; \; R_{36}^{r} f_7 = f_7 R_{36}^{10-r} \}$$

$$(R_{36}^{r} f_7)^{-1} = f_7^{-1} R_{36}^{-r}$$

$$\Rightarrow R_{36}^{r} f_7 = f_7 R_{36}^{n-r}$$

Rot :- $R_0, \; R_{1\cdot\frac{360}{10}}, \; R_{2\cdot\frac{360}{10}}, \; R_{3\cdot\frac{360}{10}}, \; R_{7\frac{360}{10}}, \ldots, R_{9\cdot\frac{360}{10}} \}$

we can choose

$\lambda = R_{m\cdot\frac{360}{10}}$, where $\gcd(m, 10) = 1$.

$m \in U(10)$

$$\left(R_{2\cdot\frac{360}{10}}\right)^5 = R_0 \qquad = \{1, 3, 7, 9\}$$

$$\left(R_{\frac{360}{5}}\right)^5 = R_0$$

$$R_{4\cdot\frac{360}{10}} = R_{2\cdot\frac{360}{5}}$$

$$\left(R_{2\cdot\frac{360}{5}}\right)^5 = R_0$$

$$R_{5\cdot\frac{360}{10}} = \left(R_{\frac{360}{2}}\right)^2 = R_0$$

$$\left(R_{7\cdot\frac{360}{10}}\right)^{10} = R_0$$

In case of D_n, we choose,

$\alpha = R_{\frac{360}{n}}$, $R_{m\cdot\frac{360}{n}}$ where $\gcd(m,n)=1$.

No of possibilities of m is $\phi(n)$.

D_{12}, $\alpha = R_{\frac{360}{12}}, R_{5\cdot\frac{360}{12}}, R_{7\cdot\frac{360}{12}}, R_{11\cdot\frac{360}{12}}$

In D_n, why reflection followed by rotation is a reflection?

why reflection followed by reflection is a rotation?

$\Rightarrow \quad D_n = \{\alpha, \beta : \alpha^n = R_0 = \beta^2 ;$

$= \{\underbrace{\alpha, \alpha^2, \ldots, \alpha^{n-1}}_{Rot}, \underbrace{\beta, \alpha\beta, \ldots \alpha^{n-1}\beta}_{Ref}\} \quad \alpha^r \beta = \beta \alpha^{n-r} \}$

Take arbitrary reflection $\alpha^p \beta$, rotation α^q

$$\alpha^q . (\alpha^p \beta) = (\alpha^q \alpha^p)\beta \rightarrow \alpha^{q+p} . \beta \quad (ref)$$

$$(\alpha^p \beta)\alpha^q = (\beta \ \alpha^{n-p})\alpha^q$$

$$= \beta . \alpha^{n-p+q} \quad (ref)$$

$$\alpha^p (\beta \alpha^q) = \alpha^p . (\alpha^{n-q}\beta)$$

$$= (\alpha^p \alpha^{n-q})\beta \quad [Asso]$$

$$= \alpha^{p+n-q}\beta$$

$$\beta . \alpha^{n-p+q} = \beta . \alpha^{-p+q}$$

$$= \alpha^{n-(-p+q)}\beta$$

$$= \alpha^{n+p-q}\beta$$

D_{12}, $p = 3$, $q = 7$

$$\alpha^{p+n-q}\beta = \alpha^8 \beta = \beta \alpha^4$$

$$= \beta \alpha^{q-p}$$

$$\alpha^r \beta = \beta \alpha^{n-r} = \beta \alpha^{-r}$$

Take two arbitrary r.c.f.

$$\alpha^{p}\beta \;,\; \alpha^{q}\beta$$

$$(\alpha^{p}\beta)(\alpha^{q}\beta) = \alpha^{p}(\beta\alpha^{q})\beta$$

$$= \alpha^{p}\alpha^{n-q}\beta\beta$$

$$= \alpha^{n+p-q}\beta^{2}$$

$$= \alpha^{n+p-q}\;(\underline{\gamma\, o\, A})$$