Subgroup test :—

Two step subgp test :—

Let $(G, \circ)$ be a group and H be a nonempty subset of G. H forms a subgroup of G iff

i) $ab \in H$, $\forall a, b \in H$

ii) $a^{-1} \in H$, $\forall a \in H$

$\Rightarrow$ Let (i) and (ii) holds.

'$\circ$' is associative in H [Heredity]

Let $a \in H \Rightarrow a^{-1} \in H$.

For $a, a^{-1} \in H$, from (i),

$a^{-1} \circ a \in H$, $a \circ a^{-1} \in H \Rightarrow e \in H$.

$\therefore (H, \circ)$ is a subgroup of $(G, \circ)$.

Converse part :—

one - step subgroup test :—

Let G be a group and H be a nonempty subset of G. H forms a subgroup of G iff $ab^{-1} \in H$, $\forall a, b \in H$.

$\Rightarrow$ Let $ab^{-1} \in H$, $\forall a, b \in H$.

Id For $a, a \in H \Rightarrow aa^{-1} \in H \Rightarrow e \in H$

Inv For $e, a \in H \Rightarrow e.a^{-1} \in H \Rightarrow a^{-1} \in H$.

$a^{-1}, a^{-1} \in H \Rightarrow (a^{-1})(a^{-1})^{-1} \in H$

$\Rightarrow a^{-1}.a \in H$

For $a, b \in H$, we have, $a, b^{-1} \in H$

$\Rightarrow \quad a.(b^{-1})^{-1} \in H$

$\Rightarrow \quad a.b \in H$ [closed].

H is associative [Heredity]

$\therefore$ H forms a subgp of G.

Conversely, let H be a subgp of G.

For $a, b \in H$, we have, $a, b^{-1} \in H$

$\Rightarrow \quad a.b^{-1} \in H$

Finite subgroup test :—

Let H be a nonempty finite subset of a group G. Then H is a subgp of G if H is closed ✓

~~H/N~~

$(\mathbb{R}^*, \cdot)$

$\{2, 3, 4, 5, \ldots 15\}$

$(\mathbb{Z}_{30}, +)$ $\{6, 12, 18, 24, 0\}$

$U(n)$ $\{$

$\langle a \rangle = \{a, a^2, \ldots, a^n = e\}$

Center of a group :— $(Z(G))$

$$Z(G) = \{a \in G : ax = xa ; x \in G\}$$

$e \in Z(G)$. $Z(G)$ is nonempty.

If G is abelian then $Z(G) = G$.

Prop:— $Z(G)$ forms a subgroup of G.

$\Rightarrow$ $e \in Z(G)$. $Z(G)$ is nonempty.

Let $a, b \in Z(G)$.

$$\left.\begin{array}{l} a.x = x.a \\ b.x = x.b \end{array}\right\} \forall x \in G. \quad \begin{array}{l} (1) \\ (2) \end{array}$$

Claim:— $ab^{-1} \in Z(G)$.

i.e. $(ab^{-1})x = x(ab^{-1})$

$b.x = x.b$

$\Rightarrow b^{-1}.(b.x) = b^{-1}.(x.b)$

$\Rightarrow x = (b^{-1}x)\,b$

$\Rightarrow xb^{-1} = b^{-1}x \longrightarrow (3)$

$(ab^{-1})x = a(b^{-1}x) = a(xb^{-1})$ [by (3)]

$= (ax)\,b^{-1}$ [Asso]

$= (xa)\,b^{-1}$ [by (1)]

$= x(ab^{-1})$.

$\Rightarrow ab^{-1} \in Z(G)$ [as x is arbitrary]

$\therefore Z(G)$ is a subgroup of G.

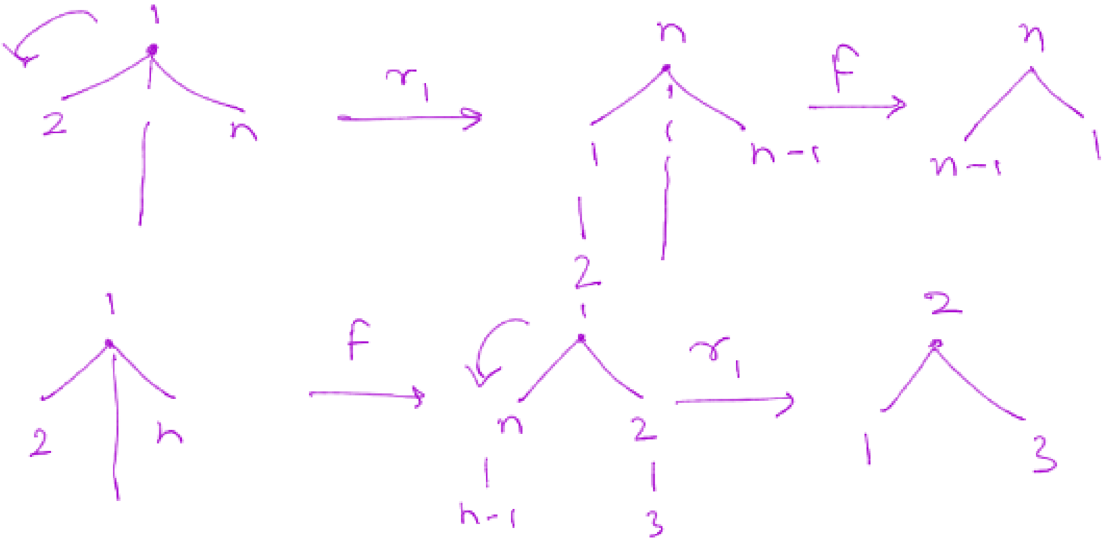
Ex:— 1) $Z(\mathbb{R}, +) = (\mathbb{R}, +)$

2) D_n, $\underline{n > 2}$

$Z(D_n) = ??$

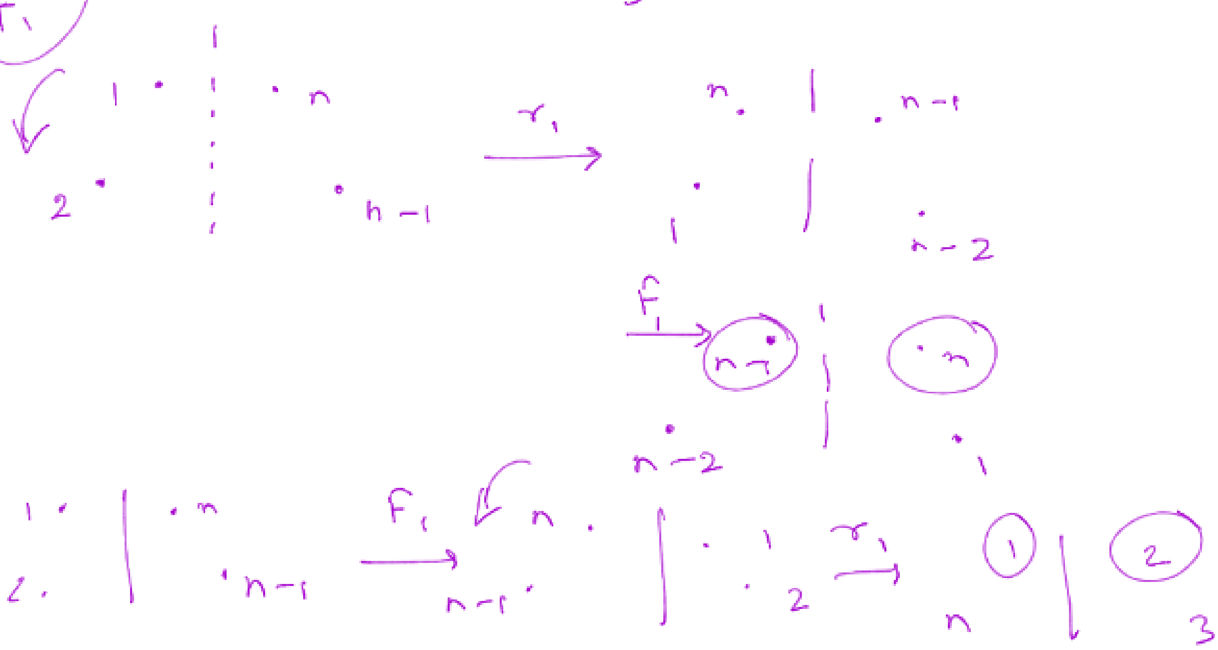
$\underline{F - \text{ref}}$, $\underline{R_{360}} = r$,

$fr_1 = r_1 f$??



$f \notin Z(D_n)$.

F_1

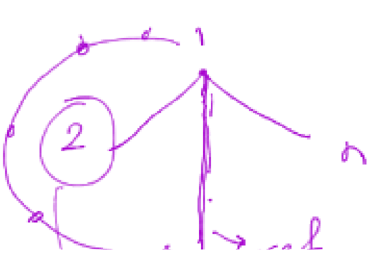


$F_1 \notin Z(D_n)$

Rot : ——— $R_0 \in Z(D_n)$

$$R_0 < R_{k \cdot \frac{360}{n}} < R_{180}$$

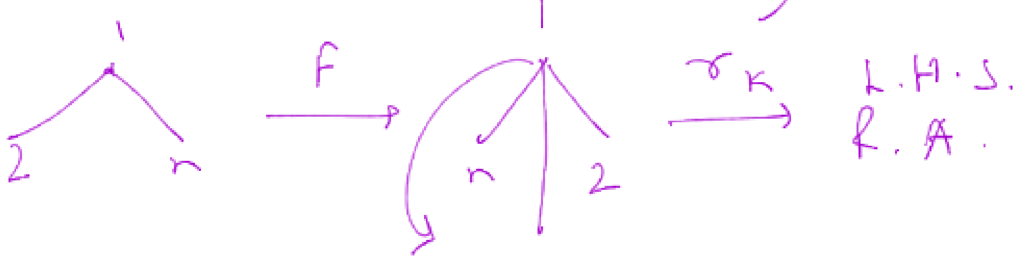
$r_k =$



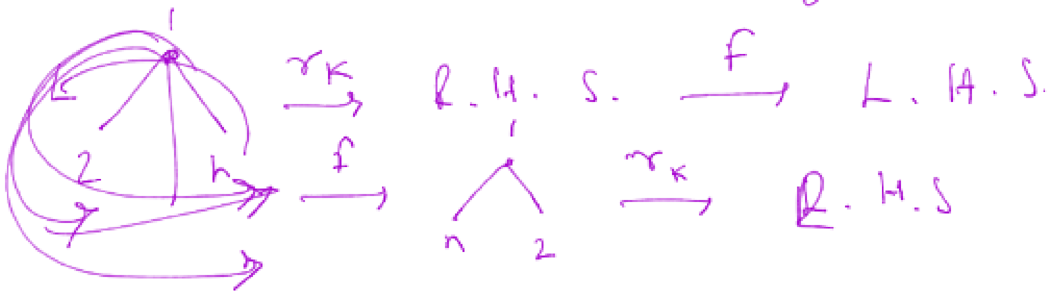
$\xrightarrow{r_k}$ L.H.S. $\xrightarrow{f}$ of R.A.

R.H.S. of R.A. (Ref. axis)

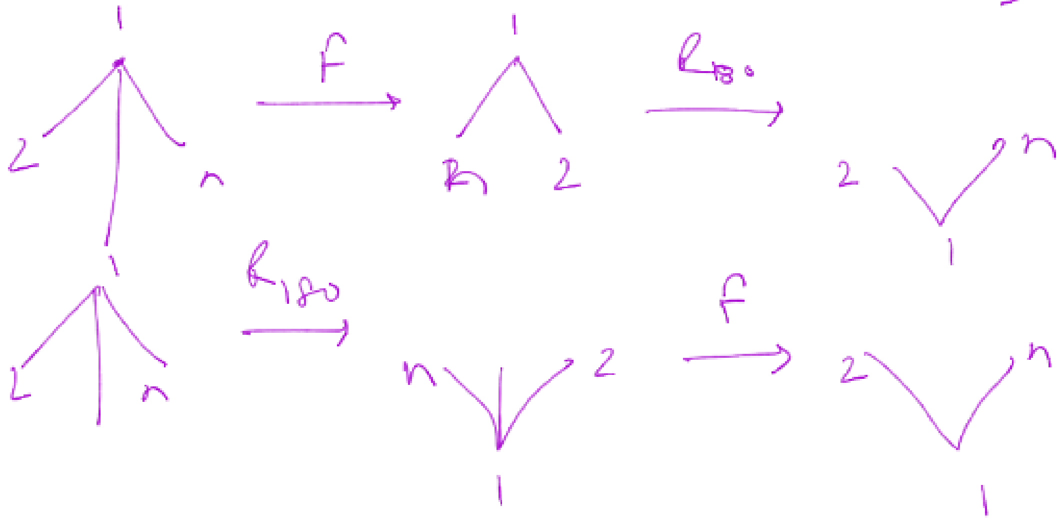
L.H.S. is (Position of 1)



$$R_{180} < R_{K \cdot \frac{360}{n}} < R_{360}$$ ✓



If n is even, $R_0, R_{180} \in Z(D_n)$



$$Z(D_n) = \begin{cases} R_0, & n \text{ is odd} \\ \{R_0, R_{180}\}, & n \text{ is even.} \end{cases}$$