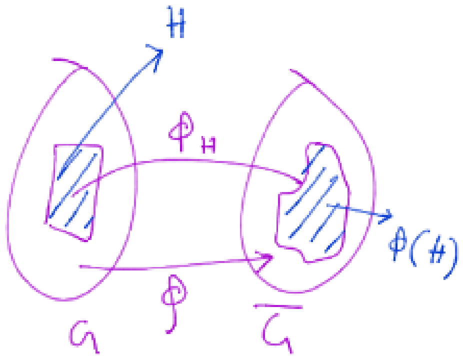
Properties of subgroups under Homomorphism:—

Let $\phi : G \to \overline{G}$ be a homomorphism, $H \leq G$. Then

i) $\phi(H) = \{\phi(h) : h \in H\}$ is a subgp of $\overline{G}$.



$\Rightarrow$ $e_{\overline{G}} \in \phi(H)$ $[\because e_G \in H]$

$\phi(H)$ is nonempty.

Let $\phi(h_1), \phi(h_2) \in \phi(H)$, $h_1, h_2 \in H$.

$\phi(h_1)\,\phi(h_2)^{-1} = \phi(h_1 h_2^{-1})$

$\in \phi(H)$ $[\because h_1 h_2^{-1} \in H]$

$\therefore$ $\phi(H) \leq \overline{G}$.

ii) If H is cyclic $\Rightarrow$ $\phi(H)$ is cyclic.

$\Rightarrow$ $H = \langle a \rangle$, $\phi(H) = \langle \phi(a) \rangle$ (claim)

Let $\phi(h_1) \in \phi(H)$

$h_1 \in H \Rightarrow h_1 = a^m, \; m \in \mathbb{Z}$

$\phi(h_1) = \phi(a^m) = \phi(a)^m$

$\therefore$ $\phi(H) = \langle \phi(a) \rangle$

iii) If H is abelian $\Rightarrow$ $\phi(H)$ is abelian.

$\phi(h_1)\,\phi(h_2) = \phi(h_1 h_2) = \phi(h_2 h_1)$

$= \phi(h_2)\,\phi(h_1)$.

iv) If $H \triangleleft G$, then $\phi(H) \triangleleft \phi(G)$

$[\overline{x h x^{-1} \in H}]$

Let $\phi(h) \in \phi(H)$, $\phi(g) \in \phi(G)$.

$$\phi(g)\,\phi(h)\,\phi(g)^{-1}$$
$$= \phi(\underline{g h g^{-1}})$$
$$\in \phi(H) \qquad \left[\, \underset{H \trianglelefteq G}{g h g^{-1} \in H} \,\right]$$

$\therefore \; \phi(H) \trianglelefteq \phi(G)$.

v) if $|\mathrm{Ker}\,\phi| = n$, then ϕ is n to 1 map from G onto $\phi(G)$.

$\Rightarrow$ for $g \in G$, $\phi(g) = g'$, then

$\phi^{-1}(g') = g\,\mathrm{Ker}\,\phi$

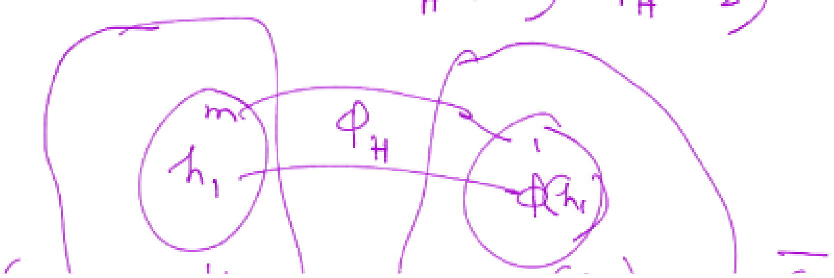
for each element in $\phi(G)$ we have n preimages.

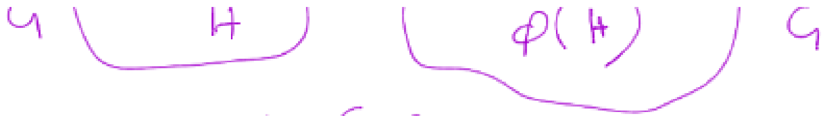
$\therefore$ ϕ is a $n-1$ map onto $\phi(G)$.

vi) if $|H| = n$, then $|\phi(H)| \,/\, n$

$\Rightarrow$ $\phi_H : H \longrightarrow \phi(H)$, ϕ_H is the restriction of ϕ on the elements of H.

$$\phi_H(h_1 h_2) = \phi(h_1 h_2)$$
$$= \phi(h_1)\,\phi(h_2)$$
$$= \phi_H(h_1)\,\phi_H(h_2)$$





$$\phi_H(h_1) = \phi(h_1)$$

$\therefore$ ϕ_H is a homomorphism from H onto $\phi(H)$.

Let $|\ker \phi_H| = m$

ϕ_H is $\underline{m \text{ to } 1}$ map.

$\therefore$

$$|\phi(H)| \cdot m = |H| = n$$

$$\Rightarrow \; |\phi(H)| \,\big/\, n.$$

vii) if $\bar{K} \leq \bar{G}$. then

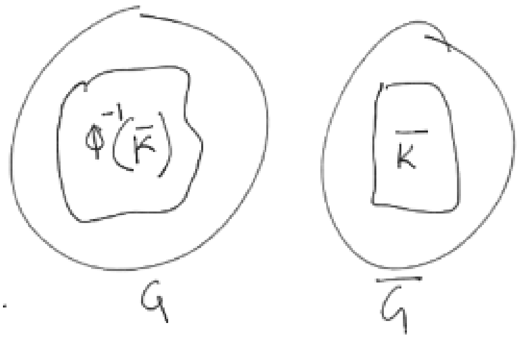
$$\phi^{-1}(\bar{K}) = \{ k \in G : \phi(k) \in \bar{K} \}$$

is a subgp of G.

$\Rightarrow$ $e_{\bar{G}} \in \bar{K}$

$\Rightarrow$ $e_G \in \phi^{-1}(\bar{K})$

$\therefore$ $\phi^{-1}(\bar{K})$ is non empty.



Let $\underline{k_1, k_2 \in \phi^{-1}(\bar{K})}$

$\phi(k_1), \phi(k_2) \in \bar{K}$

$\phi(k_2)^{-1} \in \bar{K}$ [$\because \bar{K}$ is subgp]

$$\phi(k_1 k_2^{-1}) = \phi(k_1)\,\phi(k_2)^{-1}$$

$$\in \bar{K}$$

$$\Rightarrow k_1 k_2^{-1} \in \phi^{-1}(\bar{K}).$$

$$\therefore \phi^{-1}(\bar{K}) \leq G.$$

viii) If $\bar{K} \trianglelefteq \bar{G}$ then $\phi^{-1}(\bar{K}) \trianglelefteq G$.

$\Rightarrow$ Let $k_1 \in \phi^{-1}(\bar{K})$, $g \in G$.

$$\Rightarrow \phi(k_1) \in \bar{K}$$

$$g k_1 g^{-1} \in \phi^{-1}(\bar{K}) \quad [\text{claim}]$$

$$\phi(g k_1 g^{-1}) = \underset{\in \bar{G}}{\phi(g)}\, \underset{\in \bar{K}}{\phi(k_1)}\, \underset{\in \bar{G}}{\phi(g)^{-1}}$$

$$\in \bar{K} \quad [\because \bar{K} \trianglelefteq \bar{G}]$$

$$\Rightarrow g k_1 g^{-1} \in \phi^{-1}(\bar{K}).$$

$$\therefore \phi^{-1}(\bar{K}) \trianglelefteq G.$$

ix) If ϕ is onto and $\ker\phi = \{e_G\}$.
Then ϕ is isomorphism from G to $\bar{G}$.

$\Rightarrow$ $\ker\phi = \{e_G\} \Rightarrow \phi$ is one-one.

Ex:— $\phi : \mathbb{Z}_{12} \longrightarrow \mathbb{Z}_{12}$

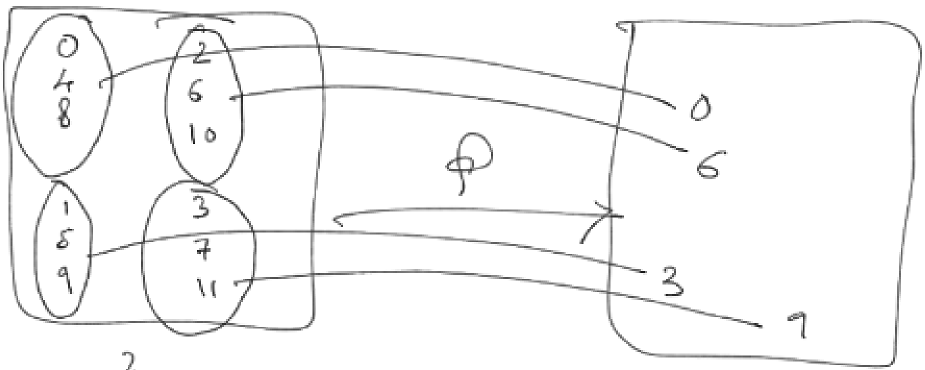
$$\phi(x) = 3x, \ \forall x.$$

$$\phi(x_1 + x_2) = 3(x_1 + x_2) = 3x_1 + 3x_2 = \phi(x_1) + \phi(x_2)$$

ϕ is Homomorphism.

$\ker\phi = \{0, 4, 8\}$, $|\ker\phi| = 3$

$\therefore \phi$ is a 3-1 map



$\mathbb{Z}_{12}$ $\qquad\qquad\qquad$ $\mathbb{Z}_{12}$

$\phi(7) = 9$

$\phi^{-1}(9) = 7 + \ker\phi = 7 + \{0, 4, 8\}$

$= \{7, 11, 3\}$

$\langle 3 \rangle$ is cyclic.

$= \{3, 6, 9, 0\}$ maps to $\{3, 6, 9, 0\}$ cyclic

$\langle 4 \rangle \xrightarrow{\text{maps}} 0$

$\langle 2 \rangle \xrightarrow{\text{maps}} \{0, 6\}$ cyclic.

$|3| = 4$, $\phi(3) = 9$

$|9| = 4$

$|2| = 6$, $\phi(2) = 6$

$|\phi(2)| = 2 / 6$.

$|4| = 3$, $\phi(4) = 0$

$|\phi(4)| = 1$

$\bar{K} = \{0, 6\}$, $\phi^{-1}(\bar{K}) = \langle 2 \rangle$

<u>Prop</u>:— $\ker\phi$ is normal in G.

$\Rightarrow$ Let $k \in \ker\phi$, $g \in G$

$$\phi(g k g^{-1}) = \phi(g)\,\phi(k)\,\phi(g)^{-1}$$
$$= e_{\bar{G}}$$
$$\Rightarrow g k g^{-1} \in \ker \phi.$$

$\ker \phi \trianglelefteq G.$