Group Homomorphism

$G, \overline{G}$ be two groups.

$\phi : G \longrightarrow \overline{G}$ preserves the group operation.

$$\phi(ab) = \phi(a)\,\phi(b), \quad \forall a, b \in G$$

$(G, \circ)$, $(\overline{G}, *)$

$$\phi(a \circ b) = \phi(a) * \phi(b), \quad \forall a, b \in G.$$

Prop :—

1) $\phi(e_G) = e_{\overline{G}}$.

2) $\phi(a^{-1}) = \phi(a)^{-1}$, $\forall a \in G$.

3) $\phi(a^n) = \phi(a)^n$, $\forall n \in \mathbb{Z}$, $\forall a \in G$.

4) Let $\phi : G \longrightarrow \overline{G}$ be an epimorphism. If G is abelian $\Rightarrow$ $\overline{G}$ is also abelian.

$\Rightarrow$ Let $\overline{a}, \overline{b} \in \overline{G}$.

$$\begin{aligned}
\overline{a}\,\overline{b} &= \phi(a)\,\phi(b) \quad [\because \phi \text{ is onto } a, b \in G] \\
&= \phi(ab) \quad [\text{Homomorphism}] \\
&= \phi(ba) \quad [\because G \text{ is abelian}] \\
&= \phi(b)\,\phi(a) = \overline{b}\,\overline{a}.
\end{aligned}$$

$\therefore$ $\overline{a}\,\overline{b} = \overline{b}\,\overline{a}$, $\forall \overline{a}, \overline{b} \in \overline{G}$.

Converse is not true :—

$G = S_3$, $\overline{G} = \{-1, 1\}$

$\phi : G \to \overline{G}$

$$\phi(\alpha) = \begin{cases} 1, & \text{when } \alpha \in A_3 \text{ (even)} \\ -1, & \alpha \notin A_3 \text{ (odd)} \end{cases}$$

ϕ is epimorphism.

$\bar{G}$ is abelian but G is not.

5) Let $\phi : G \to \bar{G}$ be an epimorphism.

If G is cyclic $\Rightarrow$ $\bar{G}$ is also cyclic.

$\Rightarrow$ $G = \langle a \rangle$

Let $\bar{b} \in \bar{G}$, so $\exists\ b \in G$ s.t. $\phi(b) = \bar{b}$

Now, $b = a^m$, $m \in \mathbb{Z}$.

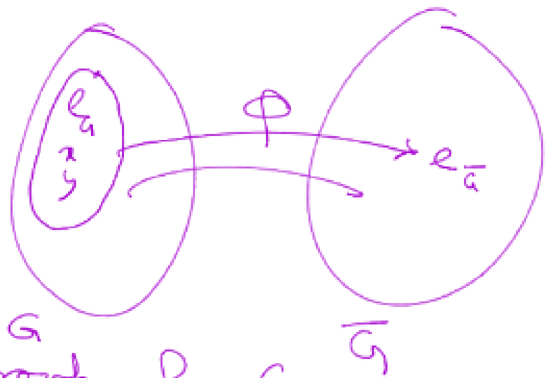
$$\bar{b} = \phi(b) = \phi(a^m) = \phi(a)^m$$

$$\bar{G} = \langle \phi(a) \rangle$$

Converse is not true :— prev. example.

Kernel of a homomorphism :—

$$\text{Ker}\,\phi = \{x \in G : \phi(x) = e_{\bar{G}}\}$$



\# Ker ϕ is a subgroup of G.

$e_G \in \text{Ker}\,\phi$, Ker ϕ is non empty.

Let $x, y \in \ker\phi$ so $\phi(x) = e_{\bar{G}} = \phi(y)$

$$\phi(xy^{-1}) = \phi(x)\,\phi(y^{-1}) \quad [\because \phi \text{ is hom}]$$

$$= \phi(x)\,\phi(y)^{-1}$$

$$= e_{\bar{G}}\, e_{\bar{G}}$$

$= e_{\bar{G}}$

$\Rightarrow xy^{-1} \in \ker\phi$

$\therefore \ker\phi$ is a subgp of G.

<u>Ex:</u>—

1) $\phi : \mathbb{R}^* \longrightarrow \mathbb{R}^*$ epimorphism.

$\phi(x) = |x| \quad \forall x \in \mathbb{R}^*$.

$\ker\phi = \{ x \in \mathbb{R}^* : \phi(x) = 1$ i.e. $|x| = 1 \}$

$= \{-1, 1\}$. $\quad$ ϕ is not one one

2) $\phi : \mathbb{Z} \longrightarrow \mathbb{Z}_n$ epimorphism

$\phi(m) = r, \quad m \equiv r \pmod{n}$

$0 \leq r < n$.

$\ker\phi = \langle n \rangle$

$= \{ x \in \mathbb{Z} : \phi(x) = \bar{0} \}$

$\mathbb{Z} \rightarrow \mathbb{Z}_9$

$7, 16, 25 \leftarrow 7$

$\ker\phi = \{-9, 0, 9, 18, \dots\}$

ϕ is not one one.

\# If ϕ is one-one iff $\ker\phi = \{e_G\}$

$\Rightarrow$ Let $\ker\phi = \{e_G\}$

Let $\phi(a) = \phi(b)$, $a, b \in G$.

$\Rightarrow \phi(ab^{-1}) = e_{\bar{G}}$

$\Rightarrow ab^{-1} \in \ker\phi$

$\Rightarrow ab^{-1} = e_G$

$\Rightarrow a = b$.

$\therefore \phi$ is one-one.

Conversely let ϕ is one-one

We know, $\phi(e_G) = e_{\bar{G}}$.

e_G is the only preimage of $e_{\bar{G}}$.

$\therefore \ker\phi = \{e_G\}$.

3) $G = GL(2, \mathbb{R})$

$\bar{G} = \mathbb{R}^*$

$|A| = 5$

pick A

A $SL(2, \mathbb{R})$

$\phi : G \to \bar{G}$

$\phi(A) = \det A, \ \forall A \in G$.

$\ker\phi = \{A \in GL(2, \mathbb{R}) : \det(A) = 1\}$

$= SL(2, \mathbb{R})$.

4) $R[x]$ = Set of all polynomials with real coeff.

$\phi : R[x] \to R[x]$

$\phi(f) = f', \ \forall f \in R[x]$.

$\ker\phi$ = constant polynomial.

Proof :—

1) $\phi : G \to \bar{G}$

Let $g \in G$.

If $|g| = n$, then $|\phi(g)| / n$.

$g^n = e_G$

$\{\phi(g)\}^n = e_{\bar{G}}$ (claim)

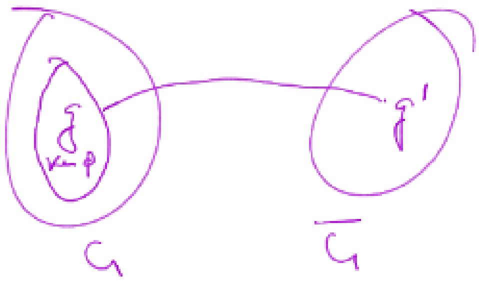
$\{\phi(g)\}^n = \phi(g^n) = \phi(e_G)$

$= e_{\bar{G}}$

$\Rightarrow |\phi(g)| \,/\, n.$

2) if $\phi(g) = g'$, then

$$\phi^{-1}(g') = \{x \in G : \phi(x) = g'\}$$
$$= g \ker\phi$$



$\phi : U(30) \to U(30)$ be a homomorphism.

$\ker\phi = \{1, 11\}$

$\phi(7) = 7$. find all elements of $U(30)$ that map to 7.

$$\phi^{-1}(7) = 7\ker\phi$$
$$= \{7, 77\}$$
$$= \{7, 17\}$$

If $\phi(7) = 13$

$$\phi^{-1}(13) = 7\ker\phi$$
$$= \{7, 17\}$$

Proof of prop 2 :—

i) $\phi^{-1}(g') \subseteq g\ker\phi$

ii) $g\ker\phi \subseteq \phi^{-1}(g')$

i) $x \in \phi^{-1}(g')$

$\Rightarrow \phi(x) = g' = \phi(g)$

$\Rightarrow \phi(g)^{-1}\phi(x) = e_{\overline{G}}$

$\Rightarrow \phi(g^{-1}x) = e_{\overline{G}}$

$\Rightarrow g^{-1}x \in \ker\phi$

$\Rightarrow x \in g\ker\phi$.

$\therefore \phi^{-1}(g') \subseteq g\ker\phi \longrightarrow (1)$

ii) Let $k \in \ker\phi \quad \therefore gk \in g\ker\phi$

$\phi(gk) = \phi(g)\,\phi(k) = \phi(g)e_{\overline{G}}$

$= \phi(g) = g'$

$\therefore gk \in \phi^{-1}(g')$

$\therefore g\ker\phi \subseteq \phi^{-1}(g') \longrightarrow (2)$

From (1) & (2),

$\phi^{-1}(g') = g\ker\phi$.

$|\ker\phi| = 4$

g' has 4 preimage.

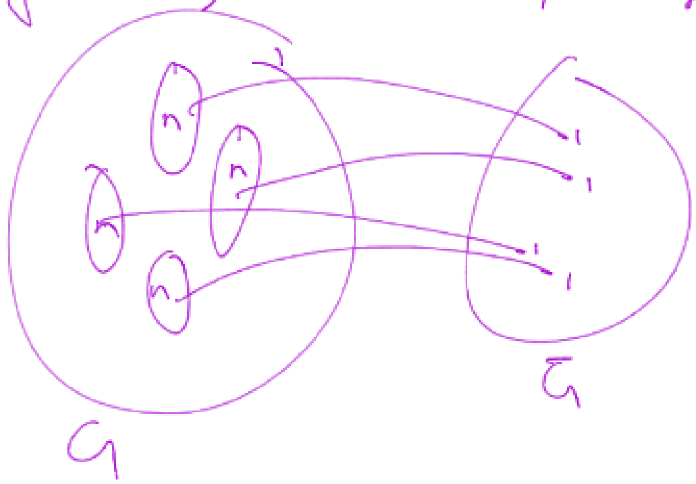
as $\phi^{-1}(g') = g\ker\phi$

ϕ is 4 to 1 map.

$|\ker\phi| = n$

Then any $g' \in \bar{G}$, has n preimages.

ϕ is a n to 1 map.



$|\ker \phi| = 1$ i.e. $\ker \phi = \{e_n\}$

ϕ is 1 to 1 map

$g' = g \ker \phi = g$