## Normal subgroup :—

$H \leq G$, $\quad aH = Ha, \ \forall a \in G$.

$H$ is normal in $G$.

$H \trianglelefteq G$.

Ex :— i) Every subgroup of an Abelian group is normal.

ii) If $[G : H] = 2$, then $H \trianglelefteq G$.

only two left cosets of $H$ in $G$, $H$ and $aH$, $a \notin H$

" " right " " ", $H$ and $Ha$, $a \notin H$

$aH = Ha = G \setminus H$.

iii) $Z(G) \trianglelefteq G$.

$a \in G$, $\quad aZ(G) = \{ax : x \in Z(G)\}$

$= \{xa : x \in Z(G)\}$

$= Z(G)a$

iv) $SL(n, \mathbb{R}) \leq GL(n, \mathbb{R})$

$A \in GL(n, \mathbb{R})$

$A\,SL(n, \mathbb{R}) = B\,SL(n, \mathbb{R})$

$\Rightarrow A^{-1}B \in SL(n, \mathbb{R})$

$\Rightarrow \det(A^{-1}B) = 1$

$\Rightarrow \det(A) = \det(B)$

Let $A \in GL(n, \mathbb{R})$

$A.SL(n, \mathbb{R}) \longrightarrow \det A$

$SL(n, \mathbb{R})A \overset{\shortparallel}{\longrightarrow} \det A$ $\quad \therefore SL(n, \mathbb{R}) \trianglelefteq GL(n, \mathbb{R})$

v) $A_n \trianglelefteq S_n$ (Using (ii))

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vi) $H$ = Subgroup of all rotations in $D_n$

$H \trianglelefteq D_n$ (Using (ii))

vii) $2\mathbb{Z} \trianglelefteq \mathbb{Z}$ (Using (i) (ii))

$m\mathbb{Z} \trianglelefteq \mathbb{Z}$ (Using (i))

viii) Trivial subgroup $\{e\}$ and improper " $G$ is normal in $G$.

<u>Simple group</u> :—

A group $G$ is said to be simple group if $G$ has no normal subgroup except the trivial and improper subgroups of $G$.

<u>Ex</u>:— Any group of prime order is simple.

<u>Thm</u> :— Let $H \leq G$.

$H \trianglelefteq G$ iff $h \in H, x \in G$

$\Rightarrow xhx^{-1} \in H$

<u>OR</u> $xHx^{-1} \subseteq H, \forall x \in G$.

$\Rightarrow$ Let $H \trianglelefteq G$.

For $h \in H$, $x \in G$,

$xh \in xH = Hx$ [$\because H \trianglelefteq G$]

$\Rightarrow xh \in Hx$

$\Rightarrow xh = h_1 x$, $h_1 \in H$

$\Rightarrow x h x^{-1} = h_1 \in H$

Conversely, let $x h x^{-1} \in H$

claim $xH \subseteq Hx$

Let $a \in xH$

$\Rightarrow a = x h_2,\ h_2 \in H$

$= x h_2 (x^{-1} x)$

$= (x h_2 x^{-1}) x$ [By Asso]

$= h_3 x$ [$\because x h x^{-1} \in H$]

$\therefore a \in Hx$

$\therefore xH \subseteq Hx$ — (1)

claim $Hx \subseteq xH$

Let $b \in Hx$

$\Rightarrow b = h_4 x,\ h_4 \in H$

$= (x x^{-1}) h_4 x$

$= x (x^{-1} h_4 x)$

$= x (x^{-1} h_4 (x^{-1})^{-1})$

$= x h_5$

$\therefore b \in xH$

$\therefore Hx \subseteq xH$ — (2)

From (1) & (2),

$xH = Hx,\ \forall x \in G.$

$\therefore H \trianglelefteq G.$

Thm: — The intersection of two normal subgroups of a group G is normal in G.

$\Rightarrow$ Let H & K be two normal

subgroups of G

subgroups of $G$.

Let $a \in H \cap K$, $x \in G$.

$a \in H$ & $a \in K$

$\therefore$ $xax^{-1} \in H$ [by prev. thm]

$xax^{-1} \in K$

$\therefore$ $xax^{-1} \in H \cap K$, $x \in G$, $a \in H \cap K$

$\therefore$ $H \cap K \trianglelefteq G$.

Thm! $\longrightarrow$ Let $H \leq G$ and $a \in G$. Then $aHa^{-1} = \{aha^{-1} : h \in H\}$ is a subgroup of $G$.

$e \in aHa^{-1}$ (Nonempty)

$ah_1a^{-1}$, $ah_2a^{-1} \in aHa^{-1}$

$$(ah_1a^{-1})(ah_2a^{-1})^{-1}$$

$$= ah_1a^{-1}(ah_2^{-1}a^{-1})$$

$$= ah_1(a^{-1}a)h_2^{-1}a^{-1}$$

$$= ah_1h_2^{-1}a^{-1}$$

$$\in aHa^{-1} \quad [\because h_1h_2^{-1} \in H]$$

Thm! $\longrightarrow$ $H \trianglelefteq G$ iff $aHa^{-1} = H$, $\forall a \in G$.

H/W