$Z_3 \oplus Z_4 = \langle (1,3) \rangle \quad |(1,3)| = \text{l.c.m.}(|1|, |3|)$

$|(a,b)| = \qquad\qquad\qquad\qquad = 12$

Thm:— The order of an element of a direct product of finite no of groups is the l.c.m. of the orders of the components of the elements.

Let $G_1, G_2, \ldots, G_n$ be the finite groups,

let $(g_1, g_2, \ldots, g_n) \in G_1 \oplus G_2 \oplus \cdots \oplus G_n$

then

$$|(g_1, g_2, \ldots, g_n)| = \text{l.c.m.}(|g_1|, |g_2|, \ldots, |g_n|)$$

$\Rightarrow \quad G_1 \oplus G_2$

$$|(g_1, g_2)| = \text{l.c.m.}(|g_1|, |g_2|)$$

Let $s = \text{l.c.m.}(|g_1|, |g_2|)$

$t = |(g_1, g_2)|$

$$(g_1, g_2)^s = (g_1^s, g_2^s) = (g_1^{mm_1}, g_2^{kk_1})$$

$|g_1| = m \qquad s = mm_1$

$|g_2| = k \qquad\; = kk_1$

$$= \left(g_1^{|g_1| m_1}, g_2^{|g_2| k_1}\right)$$

$$= (e_{G_1}, e_{G_2})$$

$\therefore \ |(g_1, g_2)| \ / \ s.$

i.e $t / s \longrightarrow$ (1)

$$(g_1^t, g_2^t) = (g_1, g_2)^t$$

$$= (e_{G_1}, e_{G_2})$$

$$g_1^t = e_{G_1}, \quad g_2^t = e_{G_2}$$

$$\Rightarrow |g_1| / t, \quad |g_2| / t$$

$$\Rightarrow \text{l.c.m.}(|g_1|, |g_2|) / t$$

$$\Rightarrow s / t \quad \text{--- (2)}$$

From (1), (2),

$$s = t.$$

Problem :—

(i) Determine the no of elements of order 5 in $\mathbb{Z}_{25} \oplus \mathbb{Z}_5$.

$$\not\cong \mathbb{Z}_{125}$$

$\mathbb{Z}_m \qquad k / m$

i) Unique cyclic subgroup of order k

ii) No of elements of order k is $\phi(k)$.

$\Rightarrow$ Let $(a, b) \in \mathbb{Z}_{25} \oplus \mathbb{Z}_5$ with $|(a, b)| = 5$

$|a| = 1,\ |b| = 5$	$|a| = 5,\ |b| = 1$	$|a| = 5 = |b|$
$b \in \mathbb{Z}_5$ (cyclic) a has one of b has four possibilities So we have four elements of order 5 here. $(0,1), (0,2), (0,3), (0,4)$	$a \in \mathbb{Z}_{25}$ (cyclic) There are 4 choices for a and 1 for b So we have 4 elements of order 5 here $(5,0), (10,0), (15,0), (20,0)$	There are 4 choices for a and b. So we have 16 elements of order 5 here. $a \in \{5, 10, 15, 20\}$ $b \in \{1, 2, 3, 4\}$

$\therefore$ There are 24 elements of order 5 in $\mathbb{Z}_{25} \oplus \mathbb{Z}_5$.

No of elements of order 25 :—

Let $(a, b) \in \mathbb{Z}_{25} \oplus \mathbb{Z}_5$ s.t.

$$|(a, b)| = 25$$

$|a| = 25,\ |b| = 1$	$|a| = 25,\ |b| = 5$
choices of a is $\phi(25)$ i.e. 20 choices of b is 1. 20 elements	choices of a is 20 " b is 4 80 elements

Total 100 elements of order 25.

$\frac{100}{\phi(25)} = 5$ cyclic subgroups of order 25

24 elements of order 5 —

$\frac{24}{\phi(5)} = 6$ cyclic subgroups of order 5.

$\langle (0,1) \rangle = \{(0,1), (0,2), (0,3), (0,4), (0,0)\}$

$\langle (5,0) \rangle = \{(5,0), (10,0), (15,0), (20,0), (0,0)\}$

$$\langle (5,1) \rangle = \{(5,1), (10,2), (15,3), (20,4), (0,0)\}$$

$$\langle (5,3) \rangle = \langle (10,1) \rangle = \{(10,1), (20,2), (5,3), (15,4), (0,0)\}$$

$$\langle (5,2) \rangle = \langle (15,1) \rangle = \{(15,1), (5,2), (20,3), (10,4), (0,0)\}$$

$$\langle (20,1) \rangle = \{(20,1), (15,2), (10,3), (5,4), (0,0)\}$$
$$= \langle (5,4) \rangle$$

H/W

$Z_{30} \oplus Z_{75}$ find the no of elements of order 5. Then find the no of cyclic subgps of order 5 and 15.

Thm:— Let G and H be two finite cyclic groups. Then $G \oplus H$ is cyclic iff g.c.d.$(|G|, |H|) = 1$.

$$Z_3 \oplus Z_5 \cong Z_{15}$$

$$Z_m \oplus Z_n \cong Z_{mn} \text{ iff } (m,n) = 1$$

$G_1 \oplus G_2 \oplus \cdots \oplus G_n$ of n no of finite cyclic groups is cyclic iff $|G_i|$ & $|G_j|$ are relatively prime when $i \neq j$.

$$Z_{m_1} \oplus Z_{m_2} \oplus \cdots \oplus Z_{m_p} \cong Z_{m_1 \cdot m_2 \cdots m_p}$$
iff $(m_i, m_j) = 1$ for $i \neq j$.

$$\mathbb{Z}_{15} \oplus \mathbb{Z}_2 \cong \mathbb{Z}_{30} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$$

$$\mathbb{Z}_m \oplus \mathbb{Z}_n \cong \mathbb{Z}_n \oplus \mathbb{Z}_m .$$

$$\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_5$$

$$\cong \mathbb{Z}_6 \oplus \mathbb{Z}_{14} \oplus \mathbb{Z}_5$$

$$\cong \mathbb{Z}_6 \oplus \mathbb{Z}_{70} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{210}$$