$\phi : G \to \bar{G}$ is an isomorphism.

1. $|a| = |\phi(a)|$, $\forall a \in G$.

Let $|a| = n$ $\quad a^n = e_G$

$\Rightarrow \phi(a^n) = e_{\bar{G}}$

$\Rightarrow \phi(a)^n = e_{\bar{G}}$

$\Rightarrow |\phi(a)| / n$. — (1)

Let $|\phi(a)| = m$.

$\phi(a)^m = e_{\bar{G}}$

$\Rightarrow \phi(a^m) = e_{\bar{G}}$

$\Rightarrow a^m = e_G$ [$\because \phi$ is one-one]

$\Rightarrow |a| / m$ — (2)

from (1) & (2), $|\phi(a)| = |a|$.

2. G is cyclic iff $\bar{G}$ is cyclic.

Let G is cyclic. $G = \langle a \rangle$.

finite: $|a| = |\phi(a)|$ $\quad \bar{G} = \langle \phi(a) \rangle$

Gen: Let $b \in G$, $b = a^k$, $k \in \mathbb{Z}$

$\phi(b) \in \bar{G}$, $\phi(b) = \phi(a^k)$

$= \phi(a)^k$, $k \in \mathbb{Z}$.

$\bar{G} = \langle \phi(a) \rangle$

Conversely $\quad \bar{G} = \langle \phi(c) \rangle$

for any $\phi(d) \in \bar{G}$,

$\phi(d) = \phi(c)^m$, $m \in \mathbb{Z}$

$\Rightarrow \phi(d) = \phi(c^m)$

$\Rightarrow \quad d = c^m$

d is arbitrary element of G.

$\therefore \quad G = \langle c \rangle$

(3) k is a fixed int.

$b \in G$, b is fixed.

(m) $x^k = b$ has the same no of soln in G as does (n) $x^k = \phi(b)$ in $\overline{G}$.

$\Rightarrow$ Let x_1 be any arbitrary solution of $x^k = b$.

$$x_1^k = b$$

$$\Leftrightarrow \quad \phi(x_1^k) = \phi(b)$$

$$\Leftrightarrow \quad \phi(x_1)^k = \phi(b) \Rightarrow y^k = \phi(b)$$

x_1 satisfies $\phi(x)^k = \phi(b)$

$$m \leq n$$

$$C^* \ncong R^*$$

$x^4 = 1 \longrightarrow 4$ solutions in C^*

$x^4 = 1 \longrightarrow 2$ solutions in $\mathbb{R}^*$.

(4) ϕ^{-1} is isomorphism from $\overline{G}$ to G.

Let $\bar{a}, \bar{b} \in \overline{G}$.

$\bar{a} = \phi(a)$, $\bar{b} = \phi(b)$, $a, b \in G$.

$$\phi^{-1}(\bar{a}\,\bar{b}) = \phi^{-1}\{\phi(a)\,\phi(b)\}$$

$$= \phi^{-1}\{\phi(ab)\}$$

$$= ab$$

$$= \phi^{-1}(\bar{a})\ \phi^{-1}(\bar{b})$$

Since $\bar{a}, \bar{b}$ are arbitrary, so

$$\phi^{-1}(\bar{a}\bar{b}) = \phi^{-1}(\bar{a})\ \phi^{-1}(\bar{b}), \forall \bar{a}, \bar{b} \in \bar{G}$$

$\therefore$ ϕ^{-1} is a homomorphism.

ϕ is bijection $\Rightarrow$ ϕ^{-1} is also bijection.

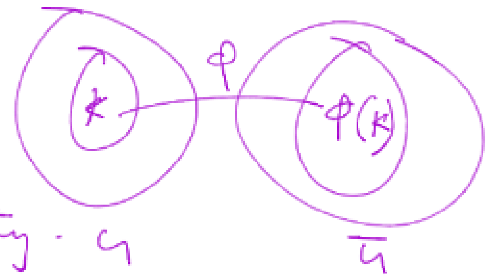
$\phi^{-1} : \bar{G} \to G$ is isomorphism.

5) $K \leq G$.

$\phi(K) = \{ \phi(k_1) : k_1 \in K \}$ is a subgp of $\bar{G}$

$\Rightarrow$ $e_G \in K$

$\phi(e_G) \in \phi(K)$



$\therefore$ $\phi(K)$ is nonempty.

Let $\phi(k_1), \phi(k_2) \in \phi(K)$, $k_1, k_2 \in K$

$$\phi(k_1) . \phi(k_2)^{-1} = \phi(k_1)\ \phi(k_2^{-1})$$

$$= \phi(k_1 k_2^{-1})$$

$k_1 k_2^{-1} \in K$, $\forall k_1, k_2 \in K$ [$\because$ K is a subgp]

$$\in \phi(K)$$

$\therefore$ $\phi(K)$ is a subgp of $\bar{G}$.

External direct product

Let G_1, G_2 be two groups.

$$G_1 \oplus G_2 = \{ (g_1, g_2) : g_1 \in G_1, g_2 \in G_2 \}$$

$$(g_1', g_2') \cdot (g_1'', g_2'') = (\underbrace{g_1' g_1''}_{G_1}, \underbrace{g_2' g_2''}_{G_2})$$

(e_{G_1}, e_{G_2}) id

operation is componentwise.

Let $G_1, G_2, \ldots, G_n$ be a finite collection of Groups.

$$G_1 \oplus G_2 \oplus \cdots \oplus G_n = \{ (g_1, g_2, \ldots, g_n) : g_i \in G_i \}$$

$$(g_1', g_2', \ldots, g_n') \cdot (g_1'', g_2'', \ldots, g_n'') = (\underbrace{g_1' g_1''}_{G_1}, \underbrace{g_2' g_2''}_{G_2}, \ldots, \underbrace{g_n' g_n''}_{G_n})$$

$(e_{G_1}, e_{G_2}, \ldots, e_{G_n})$ id

Ex:— ① $Z_2 \oplus Z_3 = \{ (0,0), (0,1), (0,2), (1,0), (1,1), (1,2) \}$

$(0,2) \cdot (1,1) = (1, 0)$

$(\bar{0}, \bar{0})$ is id.

$(a, b)(c, d) = (ac, bd)$

$(a, b) \in Z_2 \oplus Z_3$

$a^{-1} \in Z_2, \quad b^{-1} \in Z_3$

$(a, b)(a^{-1}, b^{-1}) = (0, 0)$

$|(0, 0)| = 1$

$(0,1) + (0,1) + (0,1) = (0,0)$

$$|(0,1)| = 3$$
$$|(0,2)| = 3$$
$$|(1,0)| = 2$$
$$|(1,1)| = \text{l.c.m.}(\underbrace{|1|}_{\mathbb{Z}_2}, \underbrace{|1|}_{\mathbb{Z}_3})$$
$$= \text{l.c.m.}(2, 3)$$
$$= 6.$$
$$(1,1) + (1,1) = (0,2)$$
$$3(1,1) = (1,0)$$
$$4(1,1) = (0,1)$$
$$5(1,1) = (1,2)$$
$$6(1,1) = (0,0)$$
$$\mathbb{Z}_2 \oplus \mathbb{Z}_3 = \langle (1,1) \rangle$$
$$\mathbb{Z}_2 = \langle 1 \rangle, \quad \mathbb{Z}_3 = \langle 1 \rangle = \langle 2 \rangle$$
$$\mathbb{Z}_2 \oplus \mathbb{Z}_3 = \langle (1,1) \rangle = \langle (1,2) \rangle$$
$$= \langle (\underset{\mathbb{Z}_2}{\text{gen}}, \underset{\mathbb{Z}_3}{\text{gen}}) \rangle \quad \left(\begin{array}{c}\text{In gen} \\ \text{not \underline{true}}\end{array}\right)$$
(1, 1) is a gen of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$

$(1,2) = (1,1)^{-1}$ " " " "

$$(1,1) \cdot (1,2) = (0,0)$$
$$(1,1)^{-1} = (1,2)$$

$$\mathbb{Z}_2 \oplus \mathbb{Z}_4 = \langle (1,1) \rangle \ ??$$

Is $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ cyclic ??

NO

$2(1,1) = (0,2)$

$3(1,1) = (0,2)(1,1) = (1,3)$

$4(1,1) = (1,3)(1,1) = (0,0)$

$|(1,1)| = 4 = \frac{8}{2} = \frac{\text{order of } \mathbb{Z}_2 + \mathbb{Z}_4}{\gcd(2,4)}$

$|\mathbb{Z}_2 \oplus \mathbb{Z}_4)| = 8$

$(1,1)$ is not a generator of $\mathbb{Z}_2 \oplus \mathbb{Z}_4$.

$\mathbb{Z}_m \oplus \mathbb{Z}_n$ is cyclic iff $\gcd(m,n) = 1$

$|(1,1)| = mn = \frac{\text{order of } \mathbb{Z}_m \oplus \mathbb{Z}_n}{\gcd(m,n)}$

observation :—

① $\mathbb{Z}_2 \oplus \mathbb{Z}_3 \cong \mathbb{Z}_6$

② $\mathbb{Z}_2 \oplus \mathbb{Z}_4 \not\cong \mathbb{Z}_8$ (cyclic)

(Non cyclic) both abelian.

$|\mathbb{Z}_2 \oplus \mathbb{Z}_2| = 4 = |\mathbb{Z}_4|$

$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \not\cong \mathbb{Z}_4$

both abelian.

$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \cong K\text{-}4$

$|(0,0)| = 1$ | $|(0,1)| = 2$
$|(1,0)| = 2$ | $|(1,1)| = 2$

order 8 abelian group

$\mathbb{Z}_8$, $\mathbb{Z}_4 \oplus \mathbb{Z}_2$, $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$\mathbb{Z}_8 \to$ cyclic

$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to$ 7 elements of order 2

Up to isomorphism of order atmost 8 are

$\mathbb{Z}_1 \longrightarrow$ order 1

$\mathbb{Z}_2 \longrightarrow$ " 2

$\mathbb{Z}_3 \longrightarrow$ " 3

$\left.\begin{array}{l}\mathbb{Z}_4 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2\end{array}\right\}$ abelian $\longrightarrow$ 4

$\mathbb{Z}_5 \longrightarrow$ 5

$\mathbb{Z}_6 \xrightarrow[\text{abelian}]{}$ 6

$S_3/D_3 \xrightarrow[\text{Non abelian}]{}$ 6

$\mathbb{Z}_7 \longrightarrow$ 7

$\left.\begin{array}{l}\left.\begin{array}{l}\mathbb{Z}_8 \\ \mathbb{Z}_4 \oplus \mathbb{Z}_2 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2\end{array}\right\} \text{abelian} \\ \left.\begin{array}{l}D_4 \\ Q(8)\end{array}\right\} \text{Non abelian}\end{array}\right] \longrightarrow$ of order 8.