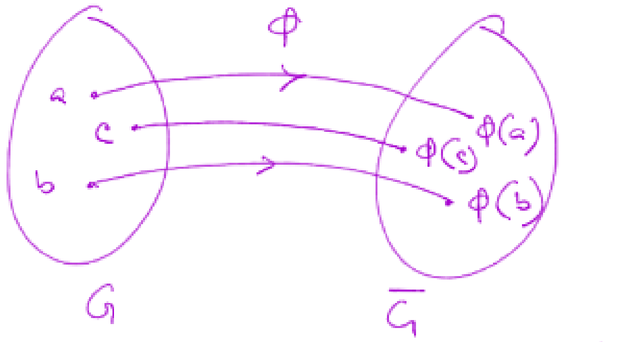
Isomorphism

$G, \bar{G}$

$$\phi : G \longrightarrow \bar{G}$$

$$\phi(ab) = \phi(a)\,\phi(b), \quad \forall a, b \in G.$$

If ϕ is bijective,
then ϕ is isomorphism.
G is isomorphic to $\bar{G}$.
i.e. $G \cong \bar{G}$



$ab = c, \quad \phi(a)\,\phi(b) = \phi(c)$

$= \phi(ab)$

Homomorphism + one one $\equiv$ Monomorphism.

Homomorphism + onto $\equiv$ Epimorphism.

Homomorphism + bijection $\equiv$ Isomorphism.

Def :—

An isomorphism $\phi : G \longrightarrow \bar{G}$, where G and $\bar{G}$ both groups, is one-one map from G onto $\bar{G}$ that preserves the group operation

$$\phi(ab) = \phi(a)\,\phi(b), \forall a, b \in G.$$

$$\phi : (G, \circ) \longrightarrow (\bar{G}, *)$$

$$\phi(a \circ b) = \phi(a) * \phi(b), \forall a, b \in G.$$

$$f : \bar{G} \rightarrow G$$

$$\phi(a) \circ \phi(b) = \phi(a * b)$$

$$\phi(a \,\cancel{*}\, b) = \phi(a)\; \cancel{*}\; \phi(b)$$

$$a \circ b = \phi^{-1}\phi(a)$$

$$\phi(a) = a', \quad \phi(b) = b'$$

$$a = \phi^{-1}(a')$$

$$\phi^{-1}(a' * b')$$

$a \circ b = \phi(a * b)$

$f : \bar{G} \to G$

$\phi^{-1}(a' * b') = \phi^{-1}(a') \circ \phi^{-1}(b')$

$f(\underline{a' * b'}) = f(a') \circ f(b')$

$\forall\, a', b' \in \bar{G}$.

$$\boxed{\text{Mapping of composition} = \text{composition of mapping}}$$

Ex:

(1) Let G be a finite cyclic group.

Let $|G| = n$. $G \cong \mathbb{Z}_n$.

$G = \langle a \rangle$, $a^n = e$

$\phi : G \longrightarrow \mathbb{Z}_n$

$\phi(a^m) = \bar{m}$, $m \in \mathbb{N}$.

Let $a^m, a^k \in G$.

$$\phi(a^m \cdot a^k) = \phi(a^{m+k})$$
$$= \overline{m+k}$$
$$= \bar{m} + \bar{k}$$
$$= \phi(a^m)\,\phi(a^k)$$

ϕ is bijection.

$\therefore G \cong \mathbb{Z}_n$

$\phi(a) = 1$

$\phi(a) = k$, $\gcd(k, n) = 1$

k is a generator of $\mathbb{Z}_n$

Let $|G| = 12$, $G = \langle a \rangle$.

$\phi : G \to \mathbb{Z}_{12}$ $\quad \phi(a) = 1$

$$\begin{cases} a \longrightarrow 1 \\ a^2 \rightarrow 2 \\ \vdots \\ a^k \rightarrow k \\ \vdots \\ a^{11} \rightarrow 11 \\ a^{12} \rightarrow 0 \end{cases}$$

$$\phi(a.b) = \phi(a) + \phi(b)$$
$$\phi(a^2) = \phi(a) + \phi(a)$$
$$= 2\phi(a) = 2$$
$$\phi(a^k) = k\,\phi(a)$$

$\phi(a) = 5$ (gen of $\mathbb{Z}_{12}$)

$$\begin{pmatrix} e & a & a^2 & a^3 & a^4 & a^5 & a^6 & a^7 & a^8 & a^9 & a^{10} & a^{11} \\ 0 & 5 & 10 & 3 & 8 & 1 & 6 & 11 & 4 & 9 & 2 & 7 \end{pmatrix}$$

$\phi(a) = 7$, $\phi(a) = 11$

Any infinite cyclic group G is isomorphic to $\mathbb{Z}$.

$G = \langle a \rangle$, $\phi : G \longrightarrow \mathbb{Z}$

$\phi(a^k) = k$, $\forall k \in \mathbb{Z}$.

2) $\phi : (\mathbb{R}, +) \longrightarrow (\mathbb{R}^+, \cdot)$

$\phi(n) = 2^n$, $\forall n \in \mathbb{R}$.

$$\phi(n_1 + n_2) = 2^{n_1 + n_2} = 2^{n_1} \cdot 2^{n_2}$$
$$= \phi(n_1) \cdot \phi(n_2).$$

3) $U(8) \cong U(12) \cong K_4$.

$\phi(3) = a \mid a(5) = a$

4) $S_3 \cong D_3$ $\quad \phi(\rho_0) = R_0$

5) $U(10) \ncong U(12)$

$\cong \mathbb{Z}$ $\quad \cong K_4$

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$$\mathbb{Z}_4 \not\cong K_4$$

6) $(\mathbb{Q}, +) \not\cong (\mathbb{Q}^*, \cdot)$

Let $\mathbb{Q} \cong \mathbb{Q}^*$ $\quad -1 \in \mathbb{Q}^*$

$$\phi(a) = -1, \text{ for } a \in \mathbb{Q}.$$

$$\phi\left(\tfrac{1}{2}a + \tfrac{1}{2}a\right) = \phi\left(\tfrac{a}{2}\right) \cdot \phi\left(\tfrac{a}{2}\right)$$

$$= \phi\left(\tfrac{a}{2}\right)^2 = -1$$

7) $G = SL(2, \mathbb{R})$

$$\phi_M : G \longrightarrow G.$$

$$\phi_M(A) = M A M^{-1}, \ \forall A \in G.$$

$$M \in G.$$

$$\phi_M(A_1 A_2) = M A_1 A_2 M^{-1}$$

$$= M A_1 (M^{-1} M) A_2 M^{-1}$$

$$= (M A_1 M^{-1})(M A_2 M^{-1})$$

$$= \phi_M(A_1)\, \phi_M(A_2).$$

ϕ_M is Automorphism.

<u>Cayley's theorem</u>:—

Every group is isomorphic to a group of permutation.

$\Rightarrow$ Let G be any group.

For any $g \in G$, define a function

$T_g : G \longrightarrow G$ by $T_g(x) = gx, \forall x \in G.$

$$\begin{pmatrix} e & x_1 & \cdots \\ g & gx_1 & \cdots \end{pmatrix}$$

T_g is a permutation on the set of elements of G.

Let $\underline{\bar{G}} = \{T_g : g \in G\}$

$U(8) = \{1, 3, 5, 7\}$

$T_1 = \begin{pmatrix} 1 & 3 & 5 & 7 \\ 1 & 3 & 5 & 7 \end{pmatrix}$, $T_3 = \begin{pmatrix} 1 & 3 & 5 & 7 \\ 3 & 1 & 7 & 5 \end{pmatrix}$

$T_7 = \begin{pmatrix} 1 & 3 & 5 & 7 \\ 7 & 5 & 3 & 1 \end{pmatrix}$ $T_5 = \begin{pmatrix} 1 & 3 & 5 & 7 \\ 5 & 7 & 1 & 3 \end{pmatrix}$

$1 \to T_1$, $3 \to T_3$, $5 \to T_5$, $7 \to T_7$

$U(8) \simeq \bar{G}$, $\underline{\bar{G}} = \{T_g : g \in U(8)\}$

$= \{T_1, T_3, T_5, T_7\}$

$\bar{G}$ is a group with function composition.

For $g, h \in G$.

$T_g T_h = T_{gh}$ (claim)

$$T_g T_h(x) = T_g(T_h(x))$$
$$= T_g(hx)$$
$$= g(hx) = (gh)x$$
$$= T_{gh}(x)$$

Since x is arbitrary,

$\therefore T_g T_h \equiv T_{gh}$

function composition is associative.

$T_g \cdot T_{e_G} = T_g = T_{e_G} T_g$,

T_{e_G} is the id of $\bar{G}$.

$T_g \cdot T_{g^{-1}} = T_e$, $(T_g)^{-1} = T_{g^{-1}}$.

$\therefore$ $\bar{G}$ is a group.

<u>claim</u> :— $G \cong \bar{G}$

Let us define a map $\phi : G \to \bar{G}$

by $\phi(g) = T_g$, $\forall g \in G$.

$g = h \Rightarrow T_g = T_h$ i.e. $\phi(g) = \phi(h)$

$\therefore$ ϕ is well defined.

$\phi(gh) = T_{gh} = T_g T_h = \phi(g)\phi(h)$

ϕ is homomorphism.

If $T_g = T_h$

$\Rightarrow T_g(e) = T_h(e)$

$\Rightarrow g = h$, one-one.

for any $T_g \in \bar{G}$, we have $g \in G$.

s.t. $\phi(g) = T_g$.

$\therefore$ ϕ is onto.

$\therefore$ ϕ is isomorphism.

$G \cong \bar{G}$.