G group. $H \le G$.

$a \, \rho_G \, b$ iff $a^{-1}b \in H$

i) $a^{-1}.a = e \in H \Rightarrow a \rho a$ (Reflexive)

ii) Let $a \rho_G b$ holds $\Rightarrow a^{-1}b \in H$

$\Rightarrow (a^{-1}b)^{-1} \in H \Rightarrow b^{-1}a \in H$

$\Rightarrow b \rho_G a$. (Sym)

iii) Let $a \rho b$ & $b \rho c$

$a^{-1}b \in H$ & $b^{-1}c \in H$

$\Rightarrow (a^{-1}b)(b^{-1}c) \in H$

$\Rightarrow a^{-1}c \in H \Rightarrow a \rho c$ (Trans)

$\therefore \rho$ is an equivalence relation on G.

$$\boxed{cl(a) = \{x : a \rho x\}}$$

Let $a \in G$.

$cl(a) = \{x : a \rho_G x \text{ holds}\}$

$\therefore a^{-1}x \in H$

$\Rightarrow a^{-1}x = h, \ h \in H$

$\Rightarrow x = ah, \ h \in H$

$cl(a) = \{ah : h \in H\}$

$a.H = \{ah : h \in H\}$

Left coset of H in G.

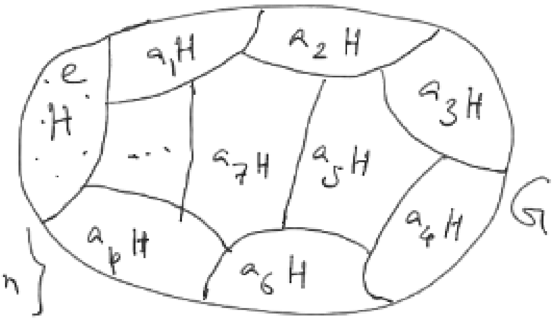
w.r.t. addition

$a + H = \{a+h : h \in H\}$

$Ha = \{ha : h \in H\}$

Right coset of H in G.

$a \in G \setminus H$



$e \in aH$??

$H = \{e, h_1, h_2, \dots, h_n\}$

$aH = \{\underset{\neq e}{a}, ah_1, ah_2, \dots, ah_n\}$

$a \neq e$, $\quad ah_i = e \Rightarrow h_i^{-1} = a \in H$ contradiction

$\therefore e \notin aH$

Is $h_j \in aH$?? , $h_j \in H$

if $h_j \in aH \Rightarrow h_j = ah_m$

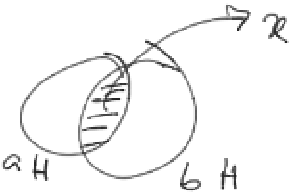
$\Rightarrow a = h_j h_m^{-1} \in H$

contradiction.

$\therefore aH \cap H = \phi$, when $a \notin H$.

$aH = bH$

$aH \cap bH = \phi$



$x \in aH \cap bH$

$\Rightarrow x \in aH$ & $x \in bH$

$\Rightarrow x = ah_1$, $x = bh_2$, $h_1, h_2 \in H$

$ah_1 = bh_2$

$\Rightarrow a^{-1}b = h_1 h_2^{-1} \in H$

$\Rightarrow a \underset{G}{\rho} b$

$\boxed{cl(a) = cl(b) \text{ iff } a \rho b}$

$\Rightarrow aH = bH$

Two left cosets are either identical or disjoint.

$$\boxed{aH = bH \text{ iff } a^{-1}b \in H}$$

$aH \neq H$, then aH is not a subgroup of G.

So, bH forms a subgroup of G

iff $bH = H$ iff $b^{-1}e \in H$

i.e. $\boxed{b \in H}$

* $\boxed{aH = H \text{ iff } a \in H}$

Let $|H| = p$, $H = \{e, a_1, a_2, \ldots, a_{p-1}\}$

$aH = \{a, a.a_1, a.a_2, \ldots, a.a_{p-1}\}$

distinct

$a.a_i = a.a_j \Rightarrow a_i = a_j$

$|aH| = p = |H|$

$G = H \cup a_1H \cup a_2H \cup \ldots \cup a_{p-1}H$

p number of disjoint left cosets with same cardinality.

$|G| = p.|H|$

$\Rightarrow \boxed{|H| \,/\, |G|.}$

Ex:— (1) $a \S b$ iff $(a-b)$ is div. by 5 in $\mathbb{Z}$.

$cl(0) = 5\mathbb{Z}$

$cl(1) = 5\mathbb{Z}+1$

$cl(2) = 5\mathbb{Z}+2$

$cl(3) = 5\mathbb{Z}+3$

$cl(4) = 5\mathbb{Z}+4$

Find all disjoint left cosets of

Find all disjoint rep.

$5\mathbb{Z}$ (subgp) in $\mathbb{Z}$ (Grp).

$a + 5\mathbb{Z}$, $a \in \mathbb{Z}$.

1) disjoint

$\bar{0} + 5\mathbb{Z} = 5\mathbb{Z} = \{\dots -10, -5, 0, 5, 10, \dots$

$\bar{1} + 5\mathbb{Z} = \{\dots, -9, -4, 1, 6, 11, \dots\}$

$\bar{2} + 5\mathbb{Z} =$

$\bar{3} + 5\mathbb{Z} =$

$\bar{4} + 5\mathbb{Z} =$

$$\mathbb{Z} = \bigcup_{a=0}^{4} a + 5\mathbb{Z}$$

Left cosets of $n\mathbb{Z}$ in $\mathbb{Z}$

$0 + n\mathbb{Z}, \dots, (n-1) + n\mathbb{Z}$.

$|G| = n$, $G = \{a_1, a_2, \dots, a_n\}$

$|H| = p$, $H = \{h_1, h_2, \dots h_p\}$

$a_1 H \cup a_2 H \cup a_3 H \cup \dots \cup a_n H = G$.

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2) Let $G = S_3$, $H = A_3 = \{(1), (123), (132)\}$

$= \{(1), (123), (132), (12), (13), (23)\}$

$(1)H = H$

$(123)H = H$

$(132)H = H$

$(12)H = \{(12), (13), (23)\}$

$= (13)H = (23)H$

$$G = H \cup (12)H$$

$aH = bH$ iff $a^{-1}b \in H$

$$(1\,2)H = (1\,3)H$$

$$(1\,2)^{-1}(1\,3) = (1\,2)(1\,3)$$
$$= (1\,3\,2) \in H$$

3) $G = D_4$, $K = \{R_0, R_{180}\}$

4) $G = D_4$, $K = \{R_0, R_{180}, H, V\}$