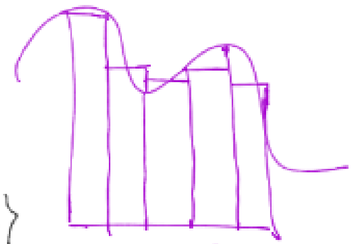
Riemann Sum :—



Let f be bounded f^n on $[a,b]$. Let $P = \{a = x_0 < x_1 < \dots < x_n = b\}$ be any partition on $[a,b]$.

$L(f,P) \le$ Riemann sum w.r.t. P $\le U(f,P)$

Then Riemann sum w.r.t. partition P is defined as $\sum_{r=1}^{n} f(\xi_r)\,\delta_r$, $\xi_r \in [x_{r-1}, x_r]$ for $r = 1(1)n$.

Thm :— Let $f : [a,b] \to \mathbb{R}$ be bounded on $[a,b]$. Then f is R-integrable on $[a,b]$ iff for every $\varepsilon > 0$, $\exists\ \delta > 0$ and for $P \in \mathbb{P}[a,b]$ with $\|P\| < \delta$ and for every choice of ξ_r in $[x_{r-1}, x_r]$ s.t.

$$\left| \sum_{r=1}^{n} f(\xi_r)\,\delta_r - \int_a^b f(x)\,dx \right| < \varepsilon .$$

$\Rightarrow$ Let f be R-int on $[a,b]$.

Then, $\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx$.

$$L \quad = \quad U \quad = \int_a^b f(x)\,dx .$$

By Darboux theorem, for given $\varepsilon > 0$, $\exists\ \delta_1, \delta_2 > 0$, s.t.

$U(f,P) < U + \varepsilon$, with $\|P\| < \delta_1$

$L(f,P) > L - \varepsilon$, with $\|P\| < \delta_2$

Let $\delta = \min(\delta_1, \delta_2)$

Then $\left.\begin{array}{l} U(f,P) < U + \varepsilon \\ L(f,P) > L - \varepsilon \end{array}\right\}$ with $\|P\| < \delta$.

$L - \varepsilon < L(f,P) < U(f,P) < U + \varepsilon$ with $\|P\| < \delta$. —— (1)

Let $P = \{a = x_0 < x_1 < \dots < x_n = b\}$.

$m_r = \inf\limits_{x \in [x_{r-1}, x_r]} f(x) \qquad M_r = \sup\limits_{x \in [x_{r-1}, x_r]} f(x)$

$r = 1(1)n$ $\qquad r = 1(1)n.$

$$m_r \le f(\xi_r) \le M_r, \quad \forall \xi_r \in [x_{r-1}, x_r]$$

$$\Rightarrow m_r \delta_r \le f(\xi_r)\delta_r \le M_r \delta_r \quad r = 1(1)n.$$

$$\Rightarrow \sum_{r=1}^{n} m_r \delta_r \le \sum_{r=1}^{n} f(\xi_r)\delta_r \le \sum_{r=1}^{n} M_r \delta_r$$

$$\Rightarrow L(f,P) \le \sum f(\xi_r)\delta_r \le U(f,P)$$

from (1)

$$L - \varepsilon < L(f,P) \le \sum_{r=1}^{n} f(\xi_r)\delta_r \le U(f,P) \le U + \varepsilon = L + \varepsilon$$

$$\Rightarrow \left| \sum_{r=1}^{n} f(\xi_r)\delta_r - L \right| < \varepsilon$$

$$\Rightarrow \left| \sum_{r=1}^{n} f(\xi_r)\delta_r - \int_a^b f(x)\,dx \right| < \varepsilon. \quad \text{with } \|P\| < \delta.$$

Conversely, let $\left| \sum f(\xi_r)\delta_r - \underbrace{\int_a^b f(x)\,dx}_{M} \right| < \varepsilon$ $\quad \underline{\underline{M \in \mathbb{R}}}$ $\quad$ with $\|P\| < \delta$.

$$\lim_{\|P\| \to 0} \sum_{r=1}^{n} f(\xi_r)\delta_r = \underbrace{\int_a^b f(x)\,dx}_{M}, \text{ for any } \xi_r \in [x_{r-1}, x_r]$$

f attains M_r for some $\xi_r \in [x_{r-1}, x_r]$

$$\lim_{\|P\| \to 0} \sum_{r=1}^{n} M_r \delta_r = \underbrace{\int_a^b f(x)\,dx}_{M}$$

$$\Rightarrow \overline{\int_a^b} f(x)\,dx = \underbrace{\int_a^b f(x)\,dx}_{M}$$

f attains m_r

$$\lim_{\|P\| \to 0} \sum_{r=1}^{n} m_r \delta_r = \underbrace{\int_a^b f(x)\,dx}_{M}$$

$$\Rightarrow \underline{\int_a^b} f(x)\,dx = \underbrace{\int_a^b f(x)\,dx}_{M}.$$

$$M = \underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx$$

$\therefore$ f is R-int

$$\lim_{\|P\| \to 0} \sum_{r=1}^{n} f(\xi_r)\,\delta_r = \int_a^b f(x)\,dx$$

N-S condition for R-integrability

$$\omega(f, P) = \sum_{r=1}^{n} (M_r - m_r)\,\delta_r \longrightarrow R$$

$$U(f, P) - L(f, P) \to 0$$

oscillatory sum $= \omega(f, P) = U(f, P) - L(f, P) \geq 0$.

Statement :— The N-S condition that a bounded function f on $[a, b]$ is R-integrable on $[a, b]$ is for $\varepsilon > 0$, $\exists\ \delta > 0$ s.t. for $P \in \mathcal{P}[a, b]$, the oscillatory sum $\omega(f, P) < \varepsilon$, $\forall P$ with $\|P\| < \delta$

i.e. $U(f, P) - L(f, P) < \varepsilon$, with $\|P\| < \delta$.

$\Rightarrow$ Let f is R-int on $[a, b]$.

Then $U = L = \int_a^b f(x)\,dx$.

By Darboux theorem, for $\varepsilon > 0$, $\exists\ \delta_1, \delta_2 > 0$

① — $U(f, P) < U + \varepsilon/2$, $\forall P$ with $\|P\| < \delta_1$

② — $L(f, P) > L - \varepsilon/2$, $\forall P$ with $\|P\| < \delta_2$

Let $\delta = \min(\delta_1, \delta_2)$

① & ② hold for $\|P\| < \delta$.

$\therefore\ L - \varepsilon/2 < L(f, P) < U(f, P) < U + \varepsilon/2$.
$= L + \varepsilon/2$, $\forall P$ with $\|P\| < \delta$.

$\Rightarrow U(f, P) - L(f, P) < (L - L) + \varepsilon/2 - (-\varepsilon/2)$
$= \varepsilon$, with $\|P\| < \delta$.

Conversely,
Let $U(f,P) - L(f,P) < \varepsilon$, with $\|P\| < \delta$.

$\Rightarrow \{U(f,P) - U\} + (U - L) + \{L - L(f,P)\} < \varepsilon$

$\Rightarrow 0 \le U - L < \varepsilon$ [$\because$ $A + B + C < \varepsilon$ where $A, B, C \ge 0$ $\Rightarrow A < \varepsilon$, $B < \varepsilon$, $C < \varepsilon$]

$\Rightarrow U - L = 0$ as ε is arbitrarily small.

$\Rightarrow U = L$

$\therefore$ f is R-int.

N-S condition - 2 :—

f is R int on $[a,b]$ iff for $\varepsilon > 0$, $\exists$ $P \in \mathcal{P}[a,b]$ s.t

$U(f,P) - L(f,P) < \varepsilon$.

$\forall P$, $\|P\| < \delta_1$

$\{x_n\}$ strictly dic

$|x_n| < \varepsilon$, $\forall n > p$

$x_n \to 0$

$|x_m| < \varepsilon$ for some $m \in \mathbb{N}$.

hold $m, m+1, m+2, \ldots$

$\forall n \ge m$.