Thm :— Let $P_1, P_2 \in P[a,b]$ with $\|P_1\| \le \delta$.
Let $f(x)$ be a real valued f^n defined on $[a,b]$ and $|f(x)| \le K$, $\forall x \in [a,b]$. Let $P_1 \subseteq P_2$ with q additional points.
Then i) $U(f,P_1) \le U(f,P_2) + 2Kq\delta$ and
ii) $L(f,P_2) \le L(f,P_1) + 2Kq\delta$

$\Rightarrow$ Let us consider P_2 with one additional point than P_1.

Let $P_1 = \{a = x_0 < x_1 < \dots < x_{r-1} < x_r < \dots < x_n = b\}$
$P_2 = \{a = x_0 < x_1 < \dots < x_{r-1} < \xi_r < x_r < \dots < x_n = b\}$

$M_r = \sup_{[x_{r-1}, x_r]} f(x)$, $r = 1(1)n$

$M_r' = \sup_{x \in [x_{r-1}, \xi_r]} f(x)$

$M_r'' = \sup_{[\xi_r, x_r]} f(x)$

$$U(f,P_1) - U(f,P_2) = (M_r - M_r')(\xi_r - x_{r-1}) + (M_r - M_r'')(x_r - \xi_r) \quad (1)$$

$|f(x)| \le K$, $\forall x \in [a,b]$

$\Rightarrow -K \le f(x) \le M_r' \le M_r \le K$, $\forall x \in [x_{r-1}, \xi_r]$

$\Rightarrow M_r - M_r' \le K - (-K)$

$\Rightarrow M_r - M_r' \le 2K$

similarly $M_r - M_r'' \le 2K$

Now from (1), $U(f,P_1) - U(f,P_2) \le 2K(x_r - x_{r-1})$
$= 2K\delta_r \le 2K\delta$ $[\because \|P_1\| \le \delta]$

$\Rightarrow U(f,P_1) \le U(f,P_2) + 2K\delta$.

Darboux theorem:—

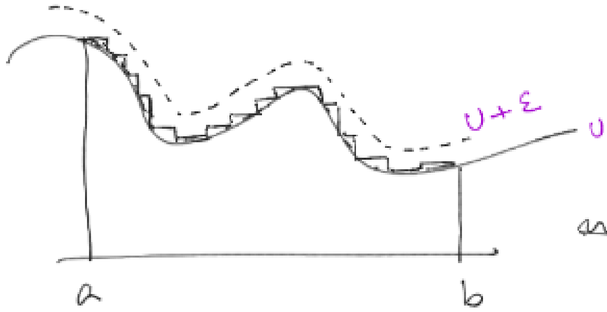
Let f be bounded f^n on $[a,b]$.
Then for $\varepsilon > 0$, $\exists\ \delta > 0$ s.t.
$U(f,P) < U + \varepsilon$, $\forall P \in P[a,b]$ with $\|P\| < \delta$.

$P_1 \subseteq P_2$

$\|P_1\| < \delta, \quad \|P_2\| \leq \delta$

$U(f, P_2) \leq U(f, P_1)$

$< U + \varepsilon$



area $\equiv U(f, P_1)$

Proof:— f is bdd on $[a,b]$. Then for $K > 0$, we have $|f(x)| \leq K, \ \forall x \in [a,b]$

$U = \operatorname{Inf}\{U(f,P) : P \in \mathcal{P}[a,b]\}$

So for given $\varepsilon > 0$, $\exists P_1 \in \mathcal{P}[a,b]$, s.t.

$$U(f, P_1) < U + \varepsilon/2 \quad \text{--- (1)}$$

Let $P_1 = \{a = x_0 < x_1 < \cdots < x_p = b\}$

Let P be any partition on $[a,b]$ with $\|P\| < \delta$, where $\delta > 0$ and

$$2K(p-1)\delta = \varepsilon/2 \quad \text{--- (2)}$$

Let P_2 be a common refinement of P_1 and P.

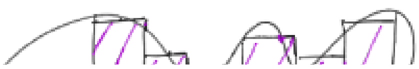
P_2 contains $(p-1)$ more points than P.

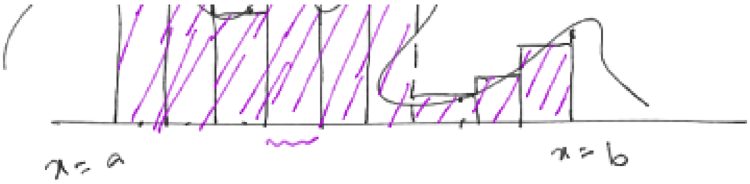
$$\begin{aligned}
U(f,P) &\leq U(f,P_2) + 2K(p-1)\delta \\
&\leq U(f,P_1) + 2K(p-1)\delta \quad [\text{Use prev. thm}] \\
&= U(f,P_1) + \varepsilon/2 \quad [\text{from (2)}] \\
&< U + \varepsilon/2 + \varepsilon/2 \quad [\text{from (1)}] \\
&= U + \varepsilon
\end{aligned}$$

$\therefore \ U(f,P) < U + \varepsilon$, $\forall P \in \mathcal{P}[a,b]$ with $\|P\| < \delta$.

H/W Darboux (Lower sum):—
$L(f,P) > L - \varepsilon, \ \forall P$ with $\|P\| < \delta$.

— x —





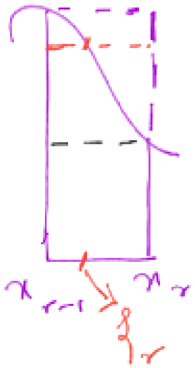
Riemann Sum

$P = \{a = x_0 < x_1 < \dots < x_{r-1} < x_r < \dots < x_n = b\}$

$\xi_r \in [x_{r-1}, x_r]$, $r = 1(1)n$.

$\sum_{r=1}^{n} f(\xi_r) \cdot \delta_r \Rightarrow$ Riemann sum.

$\delta_r = (x_r - x_{r-1})$



— $U(f, P)$
— $L(f, P)$
— Riemann sum

} in $[x_{r-1}, x_r]$

$L(f, P) \leq$ Riemann sum $\leq U(f, P)$

For $P \in \mathcal{P}[a, b]$, Lower sum $\leq$ Riemann sum $\leq$ Upper sum.