Thm :— $P_1 \subseteq P_2$, $P_1, P_2 \in \mathcal{P}[a,b]$.

$U(f, P_1) \geq U(f, P_2)$.

$L(f, P_1) \leq L(f, P_2)$.

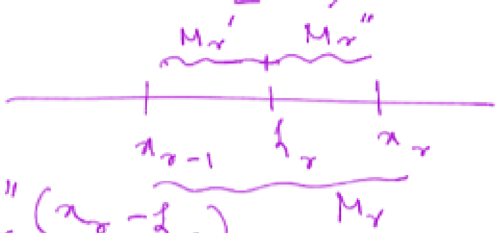
$\Rightarrow$ $P_1 = \{a = x_0 < x_1 < \dots < x_{r-1} < x_r < \dots < x_n = b\}$

$P_2 = \{a = x_0 < x_1 < \dots < x_{r-1} < \xi_r < x_r < \dots < x_n = b\}$

$P_1 \subseteq P_2$

$M_r = \sup\limits_{x \in [x_{r-1}, x_r]} f(x)$, $r = 1(1)n$, $\quad M_r' = \sup\limits_{x \in [x_{r-1}, \xi_r]} f(x)$

$M_r'' = \sup\limits_{x \in [\xi_r, x_r]} f(x)$

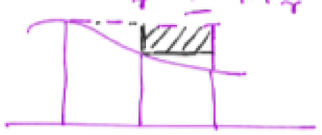


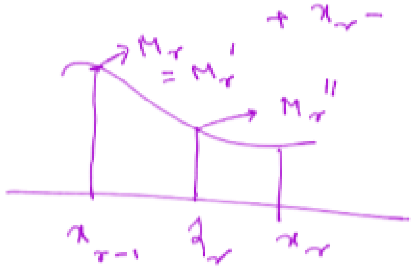
$U(f, P_1) - U(f, P_2)$

$= M_r(x_r - x_{r-1}) - M_r'(\xi_r - x_{r-1}) - M_r''(x_r - \xi_r)$

$= (M_r - M_r')(\xi_r - x_{r-1}) + (M_r - M_r'')(x_r - \xi_r)$ $\left[\because x_r - x_{r-1} = \xi_r - x_{r-1} + x_r - \xi_r\right]$

≥ 0 $\left[\begin{array}{l} M_r \geq M_r' \\ M_r \geq M_r'' \end{array}\right]$





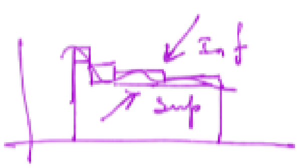
▨ $\rightarrow$ $U(f, P_1) - U(f, P_2)$

$U(f, P_1) \geq U(f, P_2)$ when $P_1 \subseteq P_2$.

Thm :— If $P_1, P_2 \in \mathcal{P}[a,b]$.

Then $L(f, P_1) \leq U(f, P_2)$.

$\Rightarrow$ Let P be a common refinement of P_1 and P_2.



$P_1 \subseteq P$, $P_2 \subseteq P$.

$L(f, P_1) \leq L(f, P) \leq U(f, P) \leq U(f, P_2)$

$\Rightarrow$ $L(f, P_1) \leq U(f, P_2)$ [use previous two thm]

Theorem :— $L \le U$ i.e. $\underline{\int}_a^b f(x)\,dx \le \overline{\int}_a^b f(x)\,dx$

$\Rightarrow$ If possible let $L > U \Rightarrow L - U > 0$

Let $\varepsilon > 0$, $L - U = 4\varepsilon > 0$ $\quad 0 < \varepsilon$

$$\Rightarrow L - U > 2\varepsilon$$

$$\Rightarrow L - \varepsilon > U + \varepsilon \quad \text{—— (A)}$$

$U = \text{Inf}\{U(f,P) : P \in \mathcal{P}[a,b]\}$

$U(f,P_1) < U + \varepsilon$, for $P_1 \in \mathcal{P}[a,b]$ (B)

$L = \text{Sup}\{L(f,P) : P \in \mathcal{P}[a,b]\}$

$L(f,P_2) > L - \varepsilon$, for $P_2 \in \mathcal{P}[a,b]$. (C)

$\therefore L(f,P_2) > L - \varepsilon > U + \varepsilon > U(f,P_1)$

$\Rightarrow L(f,P_2) > U(f,P_1)$ [from (A), (B), (C)]

for $P_1, P_2 \in \mathcal{P}[a,b]$.

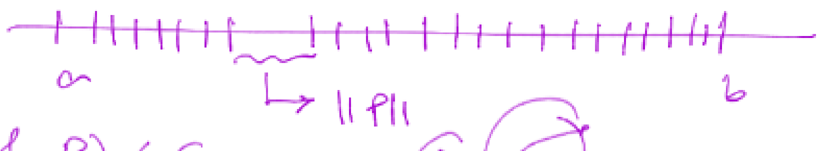
contradiction.

$\therefore L \le U$ i.e. $\underline{\int}_a^b f(x)\,dx \le \overline{\int}_a^b f(x)\,dx$

Norm of a partition :—

$P = \{a = x_0 < x_1 < \cdots < x_n = b\}$

$\|P\| = \text{Max}\{|x_r - x_{r-1}| : r = 1(1)n\}$



$U(f,P) - L(f,P) < \varepsilon$, $\|P\| < \delta$ $\quad$ (P) (δ)

$\|P\| \to 0$