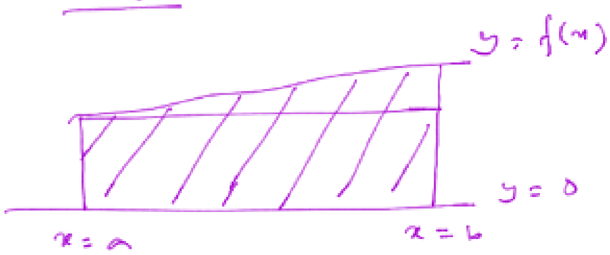
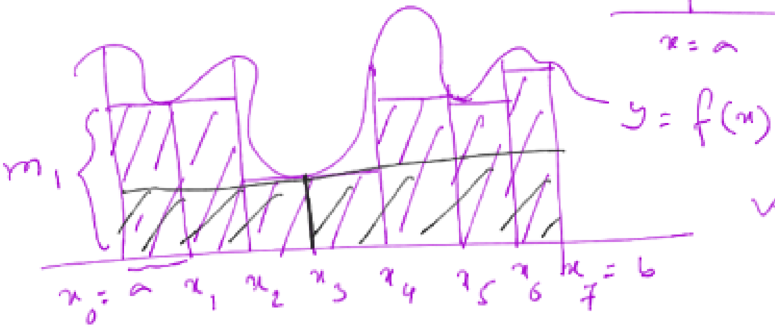
Riemann Integral

$$\int_a^b f(x)\,dx$$





$$m_1(x_1 - x_0) + m_2(x_2 - x_1) + \cdots + m_7(x_7 - x_6)$$

$$P = \{x_0 = a < x_1 < x_2 < \cdots < x_n = b\}$$

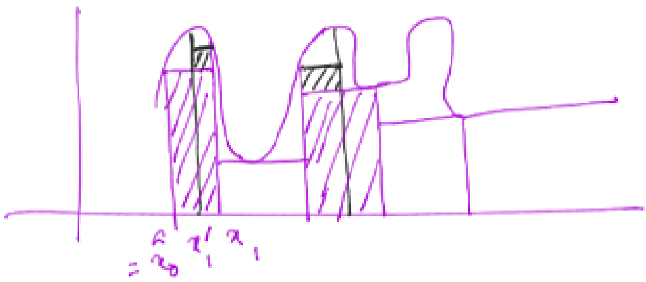
$$m_1 = \inf_{x \in [x_0, x_1]} f(x) \qquad m_r = \inf_{x \in [x_{r-1}, x_r]} f(x),$$

$r = 1(1)n.$

$$L(f, P) = \sum_{r=1}^{n} m_r (x_r - x_{r-1}) = \sum_{r=1}^{n} m_r \delta_r$$

$\delta_r = x_r - x_{r-1}$, $r = 1(1)n$.

Lower Sum of f w.r.t. partition P.



Family of Partition $\longrightarrow$ $\mathcal{P}[a, b]$.

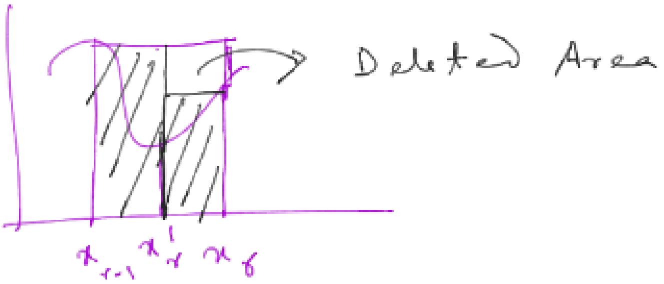
$$L = \sup\{L(f, P) : P \in \mathcal{P}[a, b]\} = \int_{\underline{a}}^{b} f(x)\,dx$$

$L \equiv$ Lower Integral.

$P = \{ a = x_0 < x_1 < \cdots \; x_n = b \}$

$M_r = \sup\limits_{x \in [x_{r-1}, x_r]} f(x) \qquad \delta_r = (x_r - x_{r-1})$

$U(f,P) = \sum\limits_{r=1}^{n} M_r \delta_r$ Upper sum of $f(x)$ w.r.t. the partition P.



$$U = \operatorname{Inf} \{ U(f,P) : P \in \mathcal{P}[a,b] \} = \overline{\int_a^b} f(x)\,dx$$

$U \equiv$ Upper integral.

* If $U = L$ i.e. $\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$

Then $f(x)$ is integrable over $[a,b]$ and

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx.$$

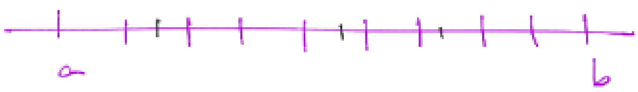
$$\left[\begin{array}{l} 0 \leq U(f,P) - L(f,P) < \varepsilon \\ 0 \leq U - L < \varepsilon \end{array} \right]$$

$\underline{P < Q}$ $\quad L(f,P) \leq L(f,Q)$

$\qquad\qquad U(f,P) \geq U(f,Q)$

Refinement :— $P = \{ a = x_0 < x_1 \ldots < x_{r-1} < x_r < \cdots < x_n = b \}$

$Q = \{ a = x_0 < x_1 < x_1' < x_2 < \cdots < x_{r-1} < x_r' < x_r < \cdots < x_n = b \}$



Here Q is a refinement of P with two additional points. x_1', x_r'.

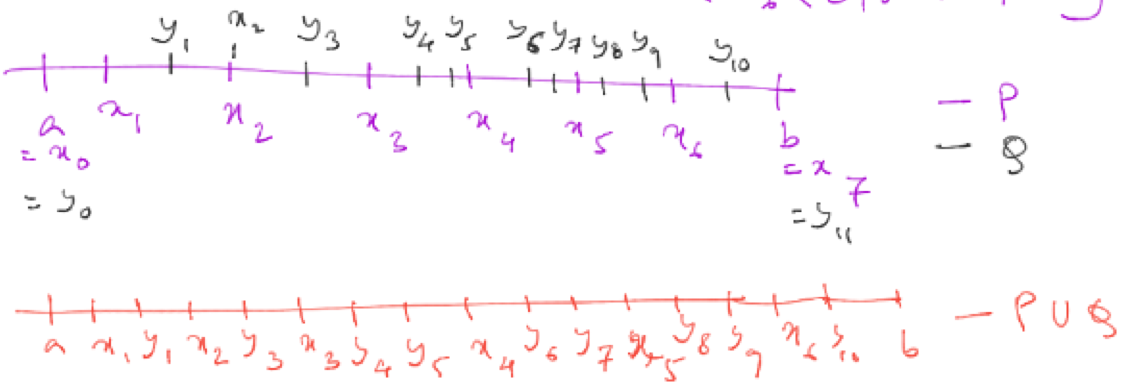
common refinement :— $P, Q \in \mathcal{P}[a,b]$

$P \cup Q$ is the smallest common refinement of P & Q.

$$P = \{a = x_0 < x_1 < x_2 < \cdots < x_7 = b\}$$

$$Q = \{a = x_0 < y_1 < x_2 < y_3 < \cdots < y_{10} < x_7 = b\}$$

$$P \cup Q = \{a = x_0 < x_1 < y_1 < x_2 < y_3 < x_3 < \cdots < x_6 < y_{10} < x_7 = b\}$$



Thm :— Let f be bounded f^n defined on $[a,b]$. If $P \in \mathcal{P}[a,b]$, then $U(f,P)$ and $L(f,P)$ both are bounded and also, $L(f,P) \le U(f,P)$, $\forall P \in \mathcal{P}[a,b]$

$\Rightarrow$ Let $P = \{a = x_0 < x_1 < \cdots < x_{r-1} < x_r < \cdots < x_n = b\}$

$$m = \inf_{x \in [a,b]} f(x), \quad M = \sup_{x \in [a,b]} f(x)$$

$$m_r = \inf_{x \in [x_{r-1}, x_r]} f(x) \qquad M_r = \sup_{x \in [x_{r-1}, x_r]} f(x)$$

$$m \le m_r \le M_r \le M, \quad \forall r = 1(1)n.$$

$$\Rightarrow m\delta_r \le m_r\delta_r \le M_r\delta_r \le M\delta_r, \quad \delta_r = x_r - x_{r-1} > 0$$

$$\Rightarrow \sum_{r=1}^{n} m\delta_r \le \sum_{r=1}^{n} m_r\delta_r \le \sum_{r=1}^{n} M_r\delta_r \le \sum_{r=1}^{n} M\delta_r$$

$$\Rightarrow m\sum_{r=1}^{n} \delta_r \le L(f,P) \le U(f,P) \le M\sum_{r=1}^{n}\delta_r$$

$$\Rightarrow m(b-a) \le L(f,P) \le U(f,P) \le M(b-a).$$

Thm :— If $P_1, P_2 \in \mathcal{P}[a,b]$ and $P_1 \subseteq P_2$ then $U(f,P_2) \le U(f,P_1)$ and $L(f,P_2) \ge L(f,P_1)$.

$\Rightarrow$ Let $P_1 = \{a = x_0 < x_1 < \cdots < x_{r-1} < x_r < \cdots < x_n = b\}$

$$P_2 = \{a = x_0 < x_1 < \cdots < x_{r-1} < \xi_r < x_r < \cdots < x_n = b\}$$

Let $M_r = \sup\limits_{x \in [x_{r-1}, x_r]} f(x)$, $r = 1(1)n$

$M_r' = \sup\limits_{x \in [x_{r-1}, \xi_r]} f(x)$, $M_r'' = \sup\limits_{x \in [\xi_r, x_r]} f(x)$

$M_r \geq M_r'$, $M_r \geq M_r''$

$$U(f, P_2) - U(f, P_1) = (M_r' - M_r)(\xi_r - x_{r-1}) + (M_r'' - M_r)(x_r - \xi_r)$$

$$\leq 0$$

$$\Rightarrow U(f, P_2) \leq U(f, P_1)$$