Proof :—

4) for $x > 0$, $L(x^n) = n L(x)$, $n \in \mathbb{Z}$.

$\Rightarrow$ $n = 0$, $L(1) = 0$

$n \in \mathbb{Z}^+$. $n = 1$, $L(x) = 1. L(x)$.

The result holds for $n = 1$.

Let the result holds for $n = k$.

i.e. $L(x^k) = k . L(x)$.

$$L(x^{k+1}) = L(x^k . x) = L(x^k) + L(x)$$

$$= k . L(x) + L(x) = (k+1) L(x)$$

So by M. I., the result holds for all (+)ve int.

Let $n \in \mathbb{Z}^-$ $\Rightarrow$ $n = -m$, $m > 0$.

$$L(x^n) = L(x^{-m}) = L\left(\left(\frac{1}{x}\right)^m\right)$$

$$= m . L\left(\frac{1}{x}\right) = -m L(x)$$

$$= n . L(x).$$

5) $L(x^\alpha) = \alpha . L(x)$, $\alpha \in \mathbb{Q}$, $x > 0$.

$\Rightarrow$ α is an int., then $L(x^\alpha) = \alpha . L(x)$

(by prop 4)

Case I :— Let $\alpha \in \mathbb{Q}^+$. $\alpha = p/q$ (say)

$p, q \in \mathbb{N}$, $q > 1$.

$$L(x^\alpha) = L(x^{p/q}) = L\{(x^{1/q})^p\}$$

$$= p . L(x^{1/q})$$

$$= \frac{p}{q} . q . L(x^{1/q})$$

$= \frac{1}{q} L(x) = \alpha \cdot L(x).$

<u>Case II</u> :— $\alpha \in \mathbb{Q}^-$, let $\alpha = -\beta$, $\beta \in \mathbb{Q}^+$

$$L(x^\alpha) = L(x^{-\beta}) = L\left(\left(\frac{1}{x}\right)^\beta\right)$$

$$= \beta L\left(\frac{1}{x}\right) = -\beta L(x) = \alpha L(x).$$

6) $L(x) = \int_1^x \frac{1}{t}\, dt$, $x > 0$ is strictly increasing on $(0, \infty)$.

$\Rightarrow$ Let $0 < x_1 < x_2 < \infty$.

$$L(x_2) - L(x_1) = \int_{x_1}^{x_2} \frac{1}{t}\, dt.$$

$\frac{1}{t} > 0$ in $[x_1, x_2]$, $\frac{1}{t}$ is cont. on $[x_1, x_2]$

$\therefore \int_{x_1}^{x_2} \frac{1}{t}\, dt > 0$

$\Rightarrow L(x_2) - L(x_1) > 0 \Rightarrow L(x_2) > L(x_1)$ when $x_2 > x_1$.

$L(x)$ is strictly increasing on $(0, \infty)$.

7) $\lim_{x \to \infty} L(x) = \infty$.

$\Rightarrow$ Let $G > 0$ be a real no which is sufficiently large.

Since $L(2) > 0$, so by Archimedean prop., $\exists\, m \in \mathbb{N}$ s.t.

$$m \cdot L(2) > G$$

$$\Rightarrow L(2^m) > G.$$

Let $x > 2^m \Rightarrow L(x) > L(2^m) > G$.

$$\Rightarrow L(x) > G \quad \forall x > 2^m$$

$$\Rightarrow \lim_{x \to \infty} L(x) = +\infty.$$

8) $\lim_{x \to 0+} L(x) = -\infty.$

$\Rightarrow$ Let $G < 0$ be a real no s.t. $|G|$ is suff. large.

$L(2) > 0$, by Archimedean prop,

$\exists$ $m \in \mathbb{N}$ s.t.

$$m L(2) > -G$$

$$\Rightarrow -m L(2) < G$$

$$\Rightarrow L(2^{-m}) < G.$$

$$\Rightarrow L\left(\frac{1}{2^m}\right) < G.$$

Let $x < \frac{1}{2^m}$ $\Rightarrow L(x) < L\left(\frac{1}{2^m}\right) < G.$

$\therefore$ $L(x) < G$, $\forall x < \frac{1}{2^m}$

$$\lim_{x \to 0+} L(x) = -\infty.$$

9) $L(x) = \int_1^x \frac{1}{t}\, dt$, $x > 0$ is cont. on $(0, \infty)$.

$\Rightarrow$ Let $f(t) = \frac{1}{t}$ in $[1, a]$, $a > 1$.

$\int_1^a f(t)\,dt$ exists. $L(x)$, $x \leq a$ is the integral f^n of f, so $L(x)$ is cont. on $[1, a]$.

a is arbitrary, so $L(x)$ is cont. on $[1, \infty)$

$[1, \infty)$. —— (A)

Let $g(t) = \frac{1}{t}$, in $[b, 1]$, $1 > b > 0$

$\int_b^1 g(t)\,dt$ exists, hence $L(x)$, $b < x \leq 1$ is the integral f^n of g.

$L(x)$ is cont. on $[b, 1]$. b is arbitrary, $L(x)$ is cont. at $(0, 1]$. —— (B)

From (A), (B), $L(x)$ is cont. at $(0, \infty)$

10)

$$\frac{d}{dx} L(x) = \frac{1}{x}, \quad x > 0.$$

$\Rightarrow$ $x > 0$, Let h be s.t. $0 < |h| < x$.

For $h > 0$, $\frac{L(x+h) - L(x)}{h} = \frac{1}{h} \int_x^{x+h} \frac{1}{t}\,dt$

$$x \leq t \leq x + h$$

$$\Rightarrow \frac{1}{x+h} \leq \frac{1}{t} \leq \frac{1}{x}$$

$$\Rightarrow \int_x^{x+h} \frac{1}{x+h}\,dt \leq \int_x^{x+h} \frac{1}{t}\,dt \leq \int_x^{x+h} \frac{1}{x}\,dt$$

$$\Rightarrow \frac{h}{x+h} \leq \int_x^{x+h} \frac{1}{t}\,dt \leq \frac{h}{x}$$

$$\Rightarrow \frac{1}{x+h} \leq \frac{1}{h} \int_x^{x+h} \frac{1}{t}\,dt \leq \frac{1}{x}$$

$$\Rightarrow \frac{1}{x+h} \leq \frac{L(x+h) - L(x)}{h} \leq \frac{1}{x}$$

$$\Rightarrow \lim_{h \to 0+} \frac{1}{x+h} \leq \lim_{h \to 0+} \frac{L(x+h) - L(x)}{h} \leq \lim_{h \to 0+} \frac{1}{x}$$

$\Rightarrow \lim \quad L(x+h) - L(x)$

$$\lim_{h\to 0+} \frac{\quad}{h} = \frac{1}{x} \quad [\text{by Sandwich}]$$

|||ly for $x<0$, we have,

$$\lim_{h\to 0-} \frac{L(x+h)-L(x)}{h} = \frac{1}{x}$$

$$\therefore \lim_{h\to 0} \frac{L(x+h)-L(x)}{h} = \frac{1}{x}.$$

$$\Rightarrow \frac{d}{dx} L(x) = \frac{1}{x}, \quad x>0.$$

ii) For $x>0$, $\dfrac{x}{1+x} < L(1+x) < x$.

$\Rightarrow$ $x>0$, let $t \in [1, 1+x]$

$$1 \le t \le 1+x$$

$$\Rightarrow \frac{1}{1+x} \le \frac{1}{t} \le 1$$

$f(t) = \frac{1}{t}$ in $[1, 1+x]$, f is cont. on $[1, 1+x]$, so integrable.

$$\int_1^{1+x} \frac{1}{1+x}\,dt \le \int_1^{1+x} \frac{1}{t}\,dt \le \int_1^{1+x} dt$$

$$\Rightarrow \frac{x}{1+x} \le L(1+x) \le x$$

At $t = 1+\frac{x}{2}$, $\dfrac{1}{1+x} < \dfrac{1}{t} < 1$

$$\Rightarrow \int_1^{1+x} \frac{dt}{1+x} < \int_1^{1+x} \frac{1}{t}\,dt < \int_1^{1+x} dt$$

$$\Rightarrow \frac{x}{1+x} < L(1+x) < x, \quad x>0.$$

iii)

17)

$2 < e < 3$

H/W

$L(x) \to \int_1^{x} \frac{1}{t}\, dt$

$$\begin{cases} L(2) < L(e) \\ L(3) > L(e) \end{cases}$$