Problem :—

1) Prove that, $-\frac{1}{2} < \int_0^1 x^3 \frac{\cos 5x}{2+x^2}\,dx < \frac{1}{2}$

Weierstrass $f(x) = \frac{x^2}{2+x^2}$, $g(x) = x\cos 5x$

$$\int_0^1 f(x)\,g(x)\,dx = f(0)\int_0^{\xi} g(x)\,dx + f(1)\int_{\xi}^1 g(x)\,dx$$

FMVT $f(x) = \cos 5x$, $g(x) = \frac{x^3}{2+x^2}$, $\forall x \in [0,1]$

f, g both int and $g(x) > 0$, $\forall x \in [0,1]$

So by FMVT, $\exists\ \xi \in [0,1]$ s.t.

$$\int_0^1 f(x)\,g(x)\,dx = f(\xi)\int_0^1 g(x)\,dx$$

$$= \cos 5\xi \int_0^1 \frac{x^3}{2+x^2}\,dx$$

$$= \cos 5\xi \int_0^1 \frac{x(2+x^2) - 2x}{2+x^2}\,dx$$

$$= \cos 5\xi \left[\int_0^1 x\,dx - \int_0^1 \frac{2x}{2+x^2}\,dx\right]$$

$$= \cos 5\xi \left[\left.\frac{x^2}{2}\right|_0^1 - \left.\log(2+x^2)\right|_0^1\right]$$

$$= \cos 5\xi \left[\frac{1}{2} - \log 3 + \log 2\right]$$

$$\left|\int_0^1 \frac{x^3\cos 5x}{2+x^2}\,dx\right| = |\cos 5\xi|\,\left|\frac{1}{2} - \log 3/2\right|$$

$$\leq \left|\frac{1}{2} - \log 3/2\right|$$

$$< \frac{1}{2}$$

2) P. T. $\frac{\pi}{6} \leq \int_0^{1/2} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \leq \frac{\pi}{6}\frac{1}{\sqrt{1-\frac{k^2}{4}}}$,

$0 < k < 2$.

$\Rightarrow \quad f(x) = \frac{1}{\sqrt{1-k^2x^2}}, \quad g(x) = \frac{1}{\sqrt{1-x^2}}, \quad \forall x \in [0, \tfrac{1}{2}].$

3) Prove that, $\phi \le \int_0^\phi \frac{dx}{\sqrt{1-\sin^2\alpha \sin^2 x}} \le \frac{\phi}{\sqrt{1-\sin^2\alpha \sin^2\phi}}$

where α, ϕ are accute angles.

$\Rightarrow \quad f(x) = \frac{1}{\sqrt{1-\sin^2\alpha \sin^2 x}}, \quad \forall x \in [0, \phi]$

To show, $\phi f(0) \le \int_0^\phi f(x)\,dx \le \phi . f(\phi)$

$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$

$f'(x) = \frac{1}{2}\left(\frac{1}{1-\sin^2\alpha \sin^2 x}\right)^{3/2} \sin^2\alpha \sin 2x \ge 0$

f is M.I. on $[0, \phi]$. $\quad \forall x \in [0, \phi]$

$m = f(0), \quad M = f(\phi), \quad b - a = \phi.$

4) P.T. $\frac{1}{2} < \int_0^1 \frac{dx}{\sqrt{4-x^2+x^3}} < \pi/6$.

$f \le g \le h$

$\Rightarrow \quad 4 - x^2 + x^3 = 4 - (x^2 - x^3) \le 4$

$4 - x^2 + x^3 \ge 4 - x^2$

$\frac{1}{2} \le \frac{1}{\sqrt{4-x^2+x^3}} \le \frac{1}{\sqrt{4-x^2}}$

5) $\frac{\pi^2}{9} < \int_{\pi/6}^{\pi/2} \frac{x}{\sin x}\,dx < \frac{2\pi^2}{9}.$

$\pi/2$

Logarithmic function :—

$$L(x) = \int_1^x \frac{dt}{t} \quad , x > 0 .$$

Prop :—

1) $L(1) = 0$.

2) $L(x) > 0$, if $x > 1$

$= 0$, if $x = 1$

< 0, if $0 < x < 1$

$\Rightarrow$ $x>1$, $f(t) = \frac{1}{t}$, $t \in [1, x]$

f is cont. and $f(t) > 0$, $\forall t \in [1, x]$

$\int_1^x f(t)\,dt > 0$

$\Rightarrow L(x) > 0$.

$0 < x < 1$, $f(t) = \frac{1}{t}$, $t \in [x, 1]$

$f(t) > 0$, $\forall t \in [x, 1]$

$\int_x^1 f(t)\,dt > 0 \Rightarrow \int_x^1 \frac{1}{t}\,dt > 0$

$\Rightarrow \int_1^x \frac{1}{t}\,dt < 0$

$\Rightarrow L(x) < 0$.

$L(1) = 0$. (Proved)

3) For $x > 0$, $y > 0$, $L(xy) = L(x) + L(y)$.

$\Rightarrow$ $xy > 0$, $L(xy) = \int_1^{xy} \frac{1}{t}\,dt$

$= \int_1^x \frac{1}{t}\,dt + \int_x^{xy} \frac{1}{t}\,dt$

$= L(x) + \int^y \frac{1}{\cdot}$ [illegible]

$\int_{1} \quad xu \qquad t = xu$

$\Rightarrow dt = x\,du$

$$= L(x) + \int_1^y \frac{1}{u}\,du = L(x) + L(y).$$

\# $y = \frac{1}{x}$, $L(x) + L\left(\frac{1}{x}\right) = L(1) = 0$

$$\Rightarrow L\left(\frac{1}{x}\right) = -L(x).$$

\# $x > 0,\ y > 0,\quad L\left(\frac{x}{y}\right) = L\left(x \cdot \frac{1}{y}\right)$

$$= L(x) + L\left(\frac{1}{y}\right) = L(x) - L(y).$$