First Mean Value theorem :—

If $f:[a,b]\to\mathbb{R}$ and $g:[a,b]\to\mathbb{R}$ be both bounded and R-int on $[a,b]$ and g preserves the same sign on $[a,b]$, then $\exists$ a real no μ lying between m and M, the inf and sup of f on $[a,b]$ s.t.

$$\int_a^b f(x)\,g(x)\,dx = \mu\int_a^b g(x)\,dx.$$

$\Rightarrow$ Since f and g both R-int on $[a,b]$, so $\int_a^b f(x)\,g(x)\,dx$ exists

Case i :— Let $g(x)\geq 0$, $\forall x\in[a,b]$.

$$m\leq f(x)\leq M,\ \forall x\in[a,b]$$

$$\Rightarrow m\,g(x)\leq f(x)\,g(x)\leq M\,g(x),\ \forall x$$

$$\Rightarrow m\int_a^b g(x)\,dx\leq\int_a^b f(x)\,g(x)\,dx\leq M\int_a^b g(x)\,dx \quad \text{(1)}$$

Let
$$\mu=\begin{cases}\dfrac{\int_a^b f(x)\,g(x)\,dx}{\int_a^b g(x)\,dx}, & \text{when } \int_a^b g(x)\,dx\neq 0\\ m, & \text{when } \int_a^b g(x)\,dx=0\end{cases}$$

Then $m\leq\mu\leq M$ and in each case
$$\int_a^b f(x)\,g(x)\,dx=\mu\int_a^b g(x)\,dx.$$
as from (1), when $\int_a^b g(x)\,dx=0$, then
$$\int_a^b f(x)\,g(x)\,dx = 0.$$

Case ii :— Let $g(x)\leq 0$, $\forall x\in[a,b]$.

$$m\leq f(x)\leq M$$

$\Rightarrow \quad m\, g(x) \geq f(x) g(x) \geq M\, g(x)$

$\Rightarrow \quad m \int_a^b g(x)\,dx \geq \int_a^b f(x) g(x)\,dx \geq M \int_a^b g(x)\,dx$ —— (2)

Let $\mu = \begin{cases} \dfrac{\int_a^b f(x) g(x)\,dx}{\int_a^b g(x)\,dx} & \text{when } \int_a^b g(x)\,dx \neq 0 \\ m, & \text{when } \int_a^b g(x)\,dx = 0. \end{cases}$

Here $m \leq \mu \leq M$ as $\int_a^b g(x)\,dx \leq 0$ when $g(x) \leq 0 \; \forall x$.

and in each case

$\int_a^b f(x) g(x)\,dx = \mu \int_a^b g(x)\,dx$, as

from (2) when $\int_a^b g(x)\,dx = 0$, then

$\int_a^b f(x) g(x)\,dx = 0.$

Cor: — If f is cont. on $[a, b]$, (All conditions of FMVT hold)

then $\exists\; \xi \in [a, b]$ s.t.

$$\int_a^b f(x) g(x)\,dx = f(\xi) \int_a^b g(x)\,dx.$$

Problem: — verify F.M.V.T. for

$f(x) = x, \; g(x) = e^x, \; \forall x \in [-1, 1].$

$\int_{-1}^{1} f(x) g(x)\,dx = 2/e$

$\int_{-1}^{1} g(x)\,dx = e - 1/e$

$\int_{-1}^{1} f(x) g(x)\,dx = f(\xi) \int_{-1}^{1} g(x)\,dx$

$\Rightarrow \quad \frac{2}{e} = \xi \cdot \left(e - \frac{1}{e}\right)$

$$\Rightarrow \quad \xi = \frac{2}{e}\cdot\frac{e}{e^2-1} = \frac{2}{e^2-1} \in [-1,1]$$

$< 2/3$

$2 < e < 3$

$$\frac{2}{e^2-1} < \frac{2}{3} \qquad e > 2 \qquad e^2-1 > 3 \qquad \frac{1}{e^2-1} < \frac{1}{3}$$

Second Mean Value theorem :—
(Bonnet's form)

If i) $f:[a,b]\to\mathbb{R}$ and $\phi:[a,b]\to\mathbb{R}$ be both int. on $[a,b]$ and

ii) f is monotone decreasing and non-neg. on $[a,b]$, then $\exists$ a point ξ in $[a,b]$ s.t.

$$\int_a^b f(x)\,\phi(x)\,dx = f(a)\int_a^{\xi}\phi(x)\,dx.$$

(Weierstrass' form)

If i) $f:[a,b]\to\mathbb{R}$ and $\phi:[a,b]\to\mathbb{R}$ be both int. on $[a,b]$ and

ii) f is monotone on $[a,b]$, then $\exists$ a point $\xi\in[a,b]$ s.t.

$$\int_a^b f(x)\,\phi(x)\,dx = f(a)\int_a^{\xi}\phi(x)\,dx + f(b)\int_{\xi}^{b}\phi(x)\,dx$$

Problem :—

Examine the validity of SMVT (Weierstrass) for $\int^{\pi} x^2\cos x\,dx$

$$\int_{-\pi}$$

$\Rightarrow$ $f(x) = x$, $\phi(x) = x\cos x$

Monotone , product of two cont. f^n, hence cont., int.

2) Prove that SMVT (Weierstrass) is applicable on $\int_a^b \frac{\sin x}{x}\,dx$, $0 < a < b < \infty$

Also $\left|\int_a^b \frac{\sin x}{x}\,dx\right| < \frac{4}{a}$.

$\Rightarrow$ Let $f(x) = \frac{1}{x}$, $\phi(x) = \sin x$, $\forall x \in [a,b]$.

M.D. Int.

By Weierstrass form (SMVT), $\exists$ $\xi \in [a,b]$, s.t.

$$\int_a^b \frac{\sin x}{x}\,dx = f(a)\int_a^{\xi} \sin x\,dx + f(b)\int_{\xi}^{b} \sin x\,dx$$

$$= \frac{1}{a}\left[-\cos x\right]_a^{\xi} + \frac{1}{b}\left[-\cos x\right]_{\xi}^{b}$$

$$= \frac{1}{a}\left[\cos a - \cos\xi\right] + \frac{1}{b}\left[\cos\xi - \cos b\right]$$

$$\left|\int_a^b \frac{\sin x}{x}\,dx\right| \leq \frac{2}{a} + \frac{2}{b}$$

$$< \frac{2}{a} + \frac{2}{a} \quad [\because a < b]$$

$$= \frac{4}{a}.$$

(Bonnet's) $\longrightarrow$ $\left|\int_a^b \frac{\sin x}{x}\,dx\right| < \frac{2}{a}$.