Absolute convergence :—

An improper integral $\int_a^b f(x)\,dx$ is called absolute convergent if $\int_a^b |f(x)|\,dx$ converges.

Thm :— Let $\int_a^b f(x)\,dx$ is absolutely conv. at the upper limit b, then $\int_a^b f(x)\,dx$ is itself convergent.

$$\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx.$$

H/W

Converse is not true :—

Ex :— $\int_0^\infty \frac{\sin x}{x}\,dx$

Abel's thm :—

Let $\int_a^\infty f(x)\,dx$ be convergent at the upper limit and let $\phi(x)$ be bounded and monotone in $[a, \infty)$, then $\int_a^\infty f(x)\,\phi(x)\,dx$ is convergent.

Dirichlet's thm :—

Let $f(x)$ bounded and integrable on $[a, X]$, $X > a$ and $\int_a^X f(x)\,dx$ be bounded on $[a, \infty)$. Let $\phi(x)$ is monotonically decreasing in $[a, \infty)$ and $\phi(x) \to 0$ as $x \to \infty$, then

$\int^\infty f(x)\,\phi(x)\,dx$ is convergent.

Problem: — $\int_0^\infty \frac{\sin x}{x}\,dx$ is convergent.

$\Rightarrow$ $\lim_{x \to 0} \frac{\sin x}{x} = 1$, $x = 0$ is removable.

$\int_0^1 \frac{\sin x}{x}\,dx$ is proper. (Riemann)
so conv.

Let us consider $\int_1^\infty \frac{\sin x}{x}\,dx$.

$f(x) = \sin x$, $\phi(x) = \frac{1}{x}$, $\forall x \in [1, \infty)$

$1 < X < \infty$

$$\left| \int_1^X f(x)\,dx \right| = \left| \int_1^X \sin x\,dx \right|$$

$$= \left| -\cos x \Big|_1^X \right|$$

$$= |\cos 1 - \cos X| \leq 2$$

$\therefore$ $\int_1^X \sin x\,dx$ is bounded on $[1, \infty)$.

$\phi(x)$ is M. D. on $[1, \infty)$ and $\phi(x) \to 0$ as $x \to \infty$

So by Dirichlet's test,

$\int_1^\infty \frac{\sin x}{x}\,dx$ is convergent.

$\therefore$ $\int_0^\infty \frac{\sin x}{x}\,dx$ is conv.

2) $\int_0^\infty \frac{\sin x}{x}\,dx$ is not abs. conv.

$\Rightarrow$ for $n \in \mathbb{N}$,

$\int^{n\pi} |\sin x|\,dx$

$$= \sum_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} \left| \frac{\sin x}{x} \right| dx$$

$$= \sum_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{x} dx$$

we put $x = (r-1)\pi + y$

$$= \sum_{r=1}^{n} \int_{0}^{\pi} \frac{|\sin\{(r-1)\pi + y\}|}{(r-1)\pi + y} dy$$

$$= \sum_{r=1}^{n} \int_{0}^{\pi} \frac{|\sin y|}{(r-1)\pi + y} dy$$

$$= \sum_{r=1}^{n} \int_{0}^{\pi} \frac{\sin y}{(r-1)\pi + y} dy$$

$$\geq \sum_{r=1}^{n} \int_{0}^{\pi} \frac{\sin y}{r\pi} dy$$

$$= \sum_{r=1}^{n} \left(-\frac{\cos y}{r\pi} \right) \Big|_{0}^{\pi}$$

$$= \sum_{r=1}^{n} \frac{1}{r\pi} (1+1) = \frac{2}{\pi} \sum_{r=1}^{n} \frac{1}{r}$$

We know $\sum_{n=1}^{\infty} \frac{1}{n}$ is div. (p series with $p=1$)

$$\lim_{n\to\infty} \int_{0}^{n\pi} \left| \frac{\sin x}{x} \right| dx = \infty$$

$\therefore \int_{0}^{\infty} \left| \frac{\sin x}{x} \right| dx$ is not conv.

3) $$\int_{0}^{\pi/2} \frac{\cos x}{x^n} dx$$

$x=0$ is the only pt of singularity.

Let $f(x) = \dfrac{\cos x}{x^n}$

$g(x) = \dfrac{1}{x^n}$ $\Bigg\}$ $\forall x \in [0, \pi/2]$

$$\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} (\cos x) = 1 \ (\neq 0)$$

So by L.C. test $\int_0^{\pi/2} f(x)\,dx$ & $\int_0^{\pi/2} g(x)\,dx$ both converge or diverge simultaneously.

$\int_0^{\pi/2} \frac{1}{x^n}\,dx$ will conv. iff $n<1$ [stand. int.]

$\int_0^{\pi/2} \frac{\cos x}{x^n}\,dx$ " " iff $n<1$.

4) Discuss the conv. of β.f^n.

$\Rightarrow$ $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ is proper for $m \geq 1$ & $n \geq 1$.

Let us consider two integrals

$\int_0^{1/2} x^{m-1}(1-x)^{n-1}\,dx$ & $\int_{1/2}^1 x^{m-1}(1-x)^{n-1}\,dx$

Here $x=0$ is the only pt of sing. if $m<1$

Let $f(x) = x^{m-1}(1-x)^{n-1}$

$g(x) = x^{m-1}$, $\forall x \in (0, \frac{1}{2}]$

Here $x=1$ is the only pt of disco. if $n<1$.

Let $f(x) = x^{m-1}(1-x)^{n-1}$

$\lim_{x \to 0+} \frac{f(x)}{g(x)} = 1 \quad (\neq 0)$

$\int_0^{1/2} g(x)\,dx = \int_0^{1/2} \frac{1}{x^{1-m}}\,dx$

will conv. iff $1-m < 1$

i.e. $m > 0$

$g(x) = (1-x)^{n-1} \; \forall x \in [1/2, 1]$

$\lim_{x \to 1-} \frac{f(x)}{g(x)} = 1 \; (\neq 0)$

$\int_{1/2}^{1} g(x)\,dx$ i.e.

$\int_{1/2}^{1} \frac{1}{(1-x)^{1-n}}\,dx$

will conv. iff $n > 0$

& $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ will converge iff $m > 0$ & $n > 0$.

5) Discuss the conv. of γ-f^n.

$\Rightarrow \int_0^\infty e^{-x} x^{n-1}\,dx$

Let us consider two integrals

$\int_0^1 e^{-x} x^{n-1}\,dx$ & $\int_1^\infty e^{-x} x^{n-1}\,dx$

Here $x = 0$ is the only pt of sing. iff $n < 1$. and the integral is proper for $n \geq 1$.

Let $\left.\begin{aligned} f(x) &= e^{-x} x^{n-1} \\ g(x) &= x^{n-1} \end{aligned}\right\} \forall x \in [0,1]$

$\lim_{x \to 0+} \frac{f(x)}{g(x)} = 1 \quad (\neq 0)$

$\int_0^1 g(x)\,dx$ i.e. $\int_0^1 x^{n-1}\,dx$ will conv. iff $n > 0$

By L.C test $\int_0^1 e^{-x} x^{n-1}\,dx$ will conv. iff $n > 0$.

$$\int_1^{\infty} e^{-x} x^{n-1}\, dx$$

$$e^{-x} x^{n-1} = \frac{x^{n-1}}{e^x}$$

$$= \frac{x^{n-1}}{1 + x + \frac{x^2}{2!} + \dots + \frac{x^r}{r!} + \dots}$$

$$< \frac{x^{n-1}}{\frac{x^r}{r!}} = \frac{r!}{x^{r-n+1}}$$

We can choose r so large s.t.

$$r - n + 1 > 1.$$

Then $\int_1^{\infty} \frac{1}{x^{r-n+1}}\, dx$ conv.

so, $\int_1^{\infty} \frac{r!}{x^{r-n+1}}\, dx$ "

By comparison test $\int_1^{\infty} e^{-x} x^{n-1}\, dx$ converges, $\forall n$.

$\therefore \int_0^{\infty} e^{-x} x^{n-1}\, dx$ will conv. iff $n > 0$.

μ-test :—

If $f(x)$ be integrable for $x \geq a$, then $\int_a^{\infty} f(x)\, dx$ converges if

$$\lim_{x \to \infty} x^{\mu} f(x) = \lambda, \quad \mu > 1.$$

and $\int^{\infty}$ ...

$\therefore \int_a^\infty f(x)\,dx$ diverges if

$$\underset{x\to\infty}{Lt}\; x^{\mu} f(x) = \lambda\ (\neq 0),\quad \mu \leq 1.$$

Ex :- 1) $\displaystyle\int_1^\infty \frac{dx}{x\sqrt{1+x^2}}$

$f(x) = \dfrac{1}{x\sqrt{1+x^2}}$ is +ve. for $x \geq 1$.

$$\underset{x\to\infty}{Lt}\; x^2 f(x) = \underset{x\to\infty}{Lt}\; \frac{x}{\sqrt{1+x^2}}$$

$$= \underset{x\to\infty}{Lt}\; \frac{1}{\sqrt{1+\frac{1}{x^2}}} = 1\ (=\lambda)$$

Here $\mu = 2 > 1$, so

$\displaystyle\int_1^\infty \frac{dx}{x\sqrt{1+x^2}}$ is conv.

$$x f(x) = \underset{x\to\infty}{Lt}\; \frac{1}{\sqrt{1+x^2}} = 0 \checkmark$$

No conclusion

$$x^{-1} f(x) = \frac{1}{x^2\sqrt{1+x^2}} = 0 \checkmark$$

2) $\displaystyle\int_0^\infty \frac{x^{3/2}}{5+3x^2}\,dx$

$$\underset{x\to\infty}{Lt}\; x^{1/2} f(x) = \underset{x\to\infty}{Lt}\; \frac{x^2}{5+3x^2}$$

$$= \underset{x\to\infty}{Lt}\; \frac{1}{3+5/x^2} = \frac{1}{3}\ (\neq 0)$$

Dir

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