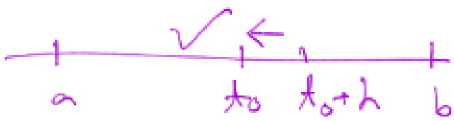
1) f R-int $\Rightarrow$ $\phi(t) = \int_a^t f(x)\,dx$, $a < t \le b$ is cont. on $[a,b]$

2) f is cont, $\phi(t) = \int_a^t f(x)\,dx$, $a < t \le b$
$\phi'(t) = f(t)$, $\forall t$.

Thm:— If f is R-int on $[a,b]$ and if $\phi(t) = \int_a^t f(x)\,dx$, $a < t \le b$ and f is cont. at $t = t_0 \in [a,b]$, then
$$\phi'(t_0) = f(t_0).$$

$\Rightarrow$
$$\left[\lim_{h \to 0+} \frac{\phi(t_0+h) - \phi(t_0)}{h} = \frac{\frac{1}{h}\int_{t_0}^{t_0+h} f(x)\,dx}{} = f(t_0) \right.$$
$$\int_{t_0}^{t_0+h} \ldots \le \frac{1}{h}\int_{t_0}^{t_0+h} f(x)\,dx \le \int_{t_0}^{t_0+h}$$
$$\left. f(t_0) - ? \le f(x) \le f(t_0) + ? \right]$$

Let $t_0 \in [a,b]$ and $h \in \mathbb{R}$ be such that $t_0 + h \in [a,b]$.



Let $M_1 = \sup_{x \in [t_0, t_0+h]} f(x)$, $m_1 = \inf_{x \in [t_0, t_0+h]} f(x)$.

$$|f(x) - f(t_0)| \le M_1 - m_1, \quad \forall x \in [t_0, t_0+h]$$
$$\Rightarrow |f(x) - f(t_0)| \le K \text{ (say)}, \quad K = M_1 - m_1$$
$$\Rightarrow f(t_0) - K \le f(x) \le f(t_0) + K.$$
$$\Rightarrow \int_{t_0}^{t_0+h} \{f(t_0) - K\}\,dx \le \int_{t_0}^{t_0+h} f(x)\,dx \le \int_{t_0}^{t_0+h} \{f(t_0) + K\}\,dx$$
$$\Rightarrow h\left[f(t_0) - K\right] \le \phi(t_0+h) - \phi(t_0) \le h\left[f(t_0) + K\right]$$

$\Rightarrow \quad f(t_0) - k \le \frac{1}{h}\{\phi(t_0+h) - \phi(t_0)\} \le f(t_0) + k$

f is cont. at t_0, $k \to 0$ as $h \to 0$

$$\lim_{h\to 0}\{f(t_0)-k\} \le \lim_{h\to 0} \frac{1}{h}\left[\phi(t_0+h) - \phi(t_0)\right]$$
$$\le \lim_{h\to 0}\{f(t_0)+k\}$$

$\Rightarrow \quad f(t_0) \le \phi'(t_0) \le f(t_0)$

$\Rightarrow \quad \phi'(t_0) = f(t_0)$.

Primitive of a function :—

Let f be a function defined on $[a,b]$. If $\exists$ a diff f^n ϕ s.t. $\phi'(x) = f(x)$, $\forall x \in [a,b]$, then ϕ is called primitive of f.

Every cont. f^n f has a primitive $\int_a^t f(x)\,dx$.
$a < t \le b$.

Ex :— 1)

$$f(x) = x, \quad x \ge 0$$
$$= -x, \quad x < 0 \quad [-5,5]$$

$\frac{5^2}{2} + \frac{5^2}{2} = 25$

$$\phi(x) = \frac{x^2}{2}, \quad x \ge 0$$
$$-\frac{x^2}{2}, \quad x < 0$$

$$\phi'(x) = f(x)$$

2) $f : [0,1] \to \mathbb{R}$

$$f(x) = 2x\cos\frac{1}{x} + \sin\frac{1}{x}, \quad x \ne 0$$
$$= 0, \quad x = 0$$

$\int_0^1 f(x)\,dx$

Sol^n 1. contd

$$\phi(x) = x^2 \cos\frac{1}{x}, \quad x \neq 0$$
$$= 0, \quad x = 0.$$

$$\therefore \phi'(x) = f(x), \quad \forall x.$$

Primitive of a f^n is not unique. as $\phi + c$ is also a primitive of f where c is any constant.

3) f is not R-int but f has a primitive.

$$f(x) = \begin{cases} 2x \sin\frac{1}{x^2} - \frac{2}{x}\cos\frac{1}{x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$\phi(x) = \begin{cases} x^2 \sin\frac{1}{x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

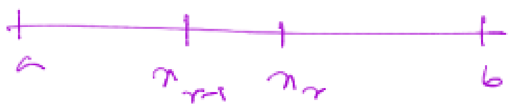
f is not bdd, so f is not R-int.

4) f is R-int, but has no primitive.

$$f(x) = x^2 \sin\frac{1}{x^2}$$

<u>Fundamental theorem of integral calculus:</u>

If a f^n $f : [a, b] \to \mathbb{R}$ be bounded and int. on $[a, b]$ and if $\exists$ a diff. f^n ϕ s.t. $\phi'(x) = f(x), \forall x \in [a, b]$ then $\int_a^b f(x)\,dx = \phi(b) - \phi(a)$.



$$\sum \phi(x_r) - \phi(x_{r-1}) = \phi'(\xi_r)(x_r - x_{r-1})$$

$$\lim_{\|P\| \to 0} \sum_{r=1} f(\xi_r)(x_r - x_{r-1})$$

$$\int_a^b f(x)\,dx$$