Improper

Comparison test :—

Let 'a' be the only point of singularity of the functions f and g which are both integrable on $[a+\varepsilon, b]$, for $0 < \varepsilon < b-a$ and

$$0 < f(x) \leq K\, g(x),\ \forall x \in (a, b],\ K > 0.$$

Then i) $\int_a^b g(x)\,dx$ is convergent

$\Rightarrow \int_a^b f(x)\,dx$ is convergent.

ii) $\int_a^b f(x)\,dx$ is divergent

$\Rightarrow \int_a^b g(x)\,dx$ is divergent.

$\Rightarrow$ i) Let $\int_a^b g(x)\,dx$ is convergent.

Then by N-S condition, $\exists$ a (+)ve real K_1, s.t. $\int_{a+\varepsilon}^b g(x)\,dx < K_1$, $0 < \varepsilon < b-a$

$$\Rightarrow \int_{a+\varepsilon}^b f(x)\,dx < K.K_1,\quad 0 < \varepsilon < b-a.$$

$$= K_2 \quad (K_2 = K.K_1)$$

$\therefore$ By N-S condition, $\int_a^b f(x)\,dx$ is converged.

ii) Let $\phi_1(\varepsilon) = \int_{a+\varepsilon}^b f(x)\,dx$

$$\phi_2(\varepsilon) = \int_{a+\varepsilon}^b g(x)\,dx$$

$$\phi_1(\varepsilon) \leq K\, \phi_2(\varepsilon),\quad 0 < \varepsilon < b-a.$$

Now $\int_a^b f(x)\,dx$ is divergent

$\Rightarrow$ $\phi_1(\varepsilon)$ is not bounded above

$\Rightarrow$ $\phi_2(\varepsilon)$ is not " "

$\Rightarrow$ $\int_a^b g(x)\,dx$ is divergent.

<u>Limit - comparison test</u> :—

Let $f(x) > 0$, $g(x) > 0$, $\forall x \in [a, b)$ except possibly a finite no of points.

Suppose $\int_a^X f(x)\,dx$ and $\int_a^X g(x)\,dx$ are proper integrals for $a < X < b$. If $\lim_{x \to b-} \frac{f(x)}{g(x)}$ exists finitely and not equal to zero, then $\int_a^b f(x)\,dx$ & $\int_a^b g(x)\,dx$ both convergent or both divergent simultaneously at the upper limit 'b'.

$\Rightarrow$ Let $\lim_{x \to b-} \frac{f(x)}{g(x)} = l \ (\neq 0)$

Now $l > 0$ as $f(x) > 0$, $g(x) > 0$, $\forall x \in [a, b)$

For given $\varepsilon > 0$, $\exists\ c$, $a < c < b$ s.t.

$$\left| \frac{f(x)}{g(x)} - l \right| < \varepsilon, \text{ when } c \leq x < b.$$

$$\Rightarrow l - \varepsilon < \frac{f(x)}{g(x)} < l + \varepsilon, \quad c \leq x < b$$

$$\Rightarrow (l - \varepsilon)\, g(x) < f(x) < (l + \varepsilon)\, g(x), \quad \text{①} \quad c \leq x < b.$$

We choose ε be s.t. $(l - \varepsilon) > 0$.

From ①, $f(x) < (l + \varepsilon)\, g(x)$, $c \leq x < b$

When $\int_c^b f(x)\,dx$ is div $\Rightarrow$ $\int_a^b g(x)\,dx$ is div

$\int_c^b g(x)\,dx$ is conv. $\Rightarrow \int_c^b f(x)\,dx$ is conv.

[by comparison test as $(l+\epsilon) > 0$, finite]

$\int_a^c f(x)\,dx$ and $\int_a^c g(x)\,dx$ is proper.

$\therefore \int_a^b f(x)\,dx$ div $\Rightarrow \int_a^b g(x)\,dx$ div. $\longrightarrow$ (A)

$\int_a^b g(x)\,dx$ conv. $\Rightarrow \int_a^b f(x)\,dx$ conv. $\longrightarrow$ (B)

Again from (1), $(l-\epsilon)g(x) < f(x)$,

$c \leq x < b$

$\int_c^b g(x)\,dx$ div $\Rightarrow \int_c^b f(x)\,dx$ div.

$\int_c^b f(x)\,dx$ conv. $\Rightarrow \int_c^b g(x)\,dx$ conv.

$\int_a^c f(x)\,dx$, $\int_a^c g(x)\,dx$ proper.

$\therefore \int_a^b g(x)\,dx$ div $\Rightarrow \int_a^b f(x)\,dx$ div. $\longrightarrow$ (C)

$\int_a^b f(x)\,dx$ conv $\Rightarrow \int_a^b g(x)\,dx$ conv. $\longrightarrow$ (d)

(Proved)

Standard integrals for comparison :—

i) $\int_a^b \frac{dx}{(b-x)^\mu}$ and $\int_a^b \frac{dx}{(x-a)^\mu}$, $-\infty < a < b < \infty$

and $\mu > 0$ are convergent iff $\mu < 1$.

$\Rightarrow \int_a^b \frac{dx}{(b-x)^\mu}$

$\ldots \quad (b-x)^{\mu}$

Here b is the only pt of singularity.

$\underline{a < x < b}$ $\quad \int_a^x \frac{dx}{(b-x)^{\mu}}$ is proper

$$= \left. \frac{(b-x)^{1-\mu}}{-(1-\mu)} \right|_a^x , \mu \neq 1$$

$$= \frac{(b-x)^{1-\mu}}{\mu-1} - \frac{(b-a)^{1-\mu}}{\mu-1}$$

$$\lim_{x \to b-} \int_a^x \frac{dx}{(b-x)^{\mu}} = -\frac{(b-a)^{1-\mu}}{\mu-1}$$

iff $1-\mu > 0$

i.e. $\mu < 1$

$\int_a^b \frac{dx}{(b-x)^{\mu}}$ converges iff $\mu < 1$.

$\underline{H/W}$

ii) $\int_a^{\infty} \frac{dx}{x^{\mu}}$, $a > 0$ is convergent iff $\mu > 1$.

$\Rightarrow \quad a < x < \infty$.

$$\int_a^x \frac{dx}{x^{\mu}} = \left. \frac{x^{1-\mu}}{1-\mu} \right|_a^x = \frac{x^{1-\mu}}{1-\mu} - \frac{a^{1-\mu}}{1-\mu}$$

$$\lim_{x \to \infty} \int_a^x \frac{dx}{x^{\mu}} = -\frac{a^{1-\mu}}{1-\mu}$$ iff $1-\mu < 0$

i.e. $\mu > 1$.

$\int_a^\infty \frac{dx}{x^\mu}$ converges iff $\mu > 1$.

Problem :— Discuss the convergence of

$$\int_0^1 \frac{x^{a-1}}{1+x}\, dx .$$

$\Rightarrow$ The integral is proper for $a \geq 1$.

When $a < 1$, 0 is the only point of singularity.

Let $f(x) = \frac{x^{a-1}}{1+x}$, $g(x) = x^{a-1}$,

$\forall x \in (0, 1]$.

$$\underset{x \to 0+}{Lt} \frac{f(x)}{g(x)} = \underset{x \to 0+}{Lt} \frac{1}{1+x} = 1 \; (\neq 0)$$

By L-C comparison, $\int_0^1 f(x)\, dx$ converges if $\int_0^1 g(x)\, dx$ i.e $\int_0^1 \frac{1}{x^{1-a}} dx$ converges iff $1 - a < 1$ i.e. $a > 0$.