Prop:- $2 < e < 3$

$\Rightarrow \quad 1 \le t \le 2 \quad \Rightarrow \frac{1}{2} \le \frac{1}{t} \le 1$

$$\int_1^2 \frac{1}{t}\,dt \le \int_1^2 dt$$

at $t = 3/2$, $\quad \frac{1}{t} < 1$

$$\int_1^2 \frac{1}{t}\,dt < \int_1^2 dt \Rightarrow L(2) < 1 = L(e)$$

$\Rightarrow \quad 2 < e \quad$ [S.M.I.]

claim

$L(3) > L(e)$

$$L(3) = \int_1^3 \frac{1}{t}\,dt = \int_1^2 \frac{1}{t}\,dt + \int_2^3 \frac{dt}{t}.$$

$$\int_1^2 \frac{1}{t}\,dt, \quad t = 2 - u, \quad 0 \le u < 1$$

$$= \int$$

$$\int_2^3 \frac{1}{t}\,dt, \quad t = 2 + u$$

$$L(3) = \int_0^1 \frac{du}{2-u} + \int_0^1 \frac{du}{2+u}$$

$$= \int_0^1 \frac{4\,du}{4-u^2}$$

$0 \le u \le 1 \quad \Rightarrow \quad \frac{1}{4-u^2} \ge \frac{1}{4}$

$$\int_0^1 \frac{1}{4-u^2}\,du \ge \int_0^1 \frac{1}{4}\,du$$

$$\Rightarrow \quad L(3) \ge \int_0^1 du = L(e)$$

at $u = \frac{1}{2}$, $\quad \frac{1}{4-u^2} > \frac{1}{4}$.

$$4\int_0^1 \frac{du}{4-u^2} > 4\int_0^1 \frac{1}{4}\,du$$

$\Rightarrow \quad L(3) > L(e) \Rightarrow 3 > e.$

$\therefore \quad 2 < e < 3.$

Improper Integral

$$\int_a^b f(t)\, dt$$

i) $f(t)$ is unbounded on $[a, b]$.

ii) $[a, b]$ is unbounded on $[a, \infty)$ or $(-\infty, b]$ or $(-\infty, \infty)$

is called Improper integral.

Defn:— A point c on $[a,b]$ is called a point of singularity of the integral $\int_a^b f(t)\,dt$ if either $c = -\infty$ or $c = \infty$ or $f(t)$ is unbounded in every nbd of c.

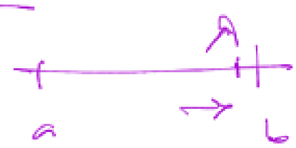
\# If b is point of singularity :—

i) If $[a,b]$ finite, $f(t)$ is unbounded in every nbd of b, then

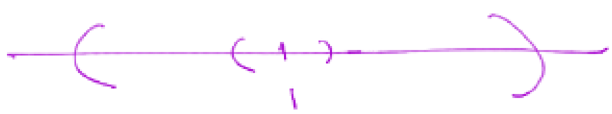
$$\int_a^b f(t)\, dt = \underset{\varepsilon \to 0+}{Lt} \int_a^{b-\varepsilon} f(t)\,dt, \quad \varepsilon > 0,\ \varepsilon < b-a$$

$$= \underset{\lambda \to b-}{Lt} \int_a^{\lambda} f(t)\,dt$$ (Provided limit exists) (P.L.E.)

$$\int_0^1 \frac{1}{1-x}\, dx$$



$x = 1$



$f_{-}(a, b) = \lim f(a+h, b) - f(a;b)$

$$\lim_{h \to 0} \frac{\quad}{h}$$

(P.L.E.)

ii) If domain is $[a, \infty)$, then

$$\int_a^{\infty} f(t)\,dt = \lim_{X \to \infty} \int_a^X f(t)\,dt, \; x > 0.$$

(P.L.E.)

\# If a is point of singularity:—

i) If $[a, b]$ finite,

$$\int_a^b f(t)\,dt = \lim_{\varepsilon \to 0+} \int_{a+\varepsilon}^b f(t)\,dt$$

$$= \lim_{\lambda \to a+} \int_{\lambda}^b f(t)\,dt \quad [\text{P.L.E.}]$$

ii) If domain is $(-\infty, b]$

$$\int_{-\infty}^b f(t)\,dt = \lim_{X \to -\infty} \int_X^b f(t)\,dt$$

(P.L.E.)

If limit does not exist, then $\int_a^b f(t)\,dt$ diverges.

If limit exists, then $\int_a^b f(t)\,dt$ converges.

\# If c be a point of singularity, where $a < c < b$.

$$\underbrace{\int_a^c f(t)\,dt}_{\text{imp}} + \underbrace{\int_c^b f(t)\,dt}_{\text{imp}}$$

$\int_a^b f(t)\,dt$ exists if both $\int_a^c f(t)\,dt$ and $\int^b f(t)\,dt$ converge and

$$\int_a^b f(t)\,dt = \int_a^c f(t)\,dt + \int_c^b f(t)\,dt$$

<u>Problem</u> :—

1) Examine the convergence :—

a) $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}}$.

Here $x=1$ is the only point of singularity.

$$\lim_{\lambda\to 1-} \int_0^\lambda \frac{dx}{\sqrt{1-x^2}} = \lim_{\lambda\to 1-} \sin^{-1}x \Big|_0^\lambda$$

$$= \lim_{\lambda\to 1-} \sin^{-1}\lambda = \pi/2.$$

If a, b both points of singularities

$$\int_a^b f(t)\,dt = \lim_{\substack{\varepsilon\to 0+\\ \delta\to 0+}} \int_{a+\varepsilon}^{b-\delta} f(t)\,dt$$

b) $\displaystyle\int_0^2 \frac{dx}{2x-x^2}$. <u>diverges</u>.

c) $\displaystyle\int_0^\infty \frac{dx}{(x+1)(x+2)} = \underline{\log 2}$

<u>Def</u> :— An integral $\int_a^b f(x)\,dx$, where f is unbounded on $[a,b]$, is called improper integral of first kind.

An integral $\int_a^\infty f(x)\,dx$ or

$\int^b \quad (\quad) \quad \int_{-\infty}$

$\int_{-\infty} f(x)\,dx$ or $\int_{-\infty} f(x)\,dx$ is called improper integral of second type/kind.

$\int_0^\infty \frac{1}{1-x}\,dx$ is of mixed types/kind.

$$\int_0^2 \frac{1}{1-x}\,dx \qquad \int_2^\infty \frac{1}{1-x}\,dx$$

$$\int_0^1 \qquad \int_1^a \qquad \int_a^\infty$$

The N-S condition :—

The N-S condition that the improper integral $\int_a^b f(x)\,dx$ converges at $x = a$, where $f(x) > 0$, $\forall x \in [a,b]$ is $\exists$ a $(+)$ve real K such that,

$$\int_{a+\varepsilon}^{b} f(x)\,dx < K, \text{ for } 0 < \varepsilon \leq b-a$$

$$\int_a^b = \int_a^c + \int_c^b \to \text{div}$$

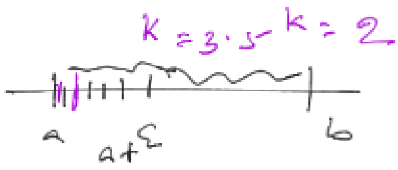
div $\not< K$ $\qquad \not< K$

$\int_{a+\varepsilon}^{b}$ div.

$\int_a^{c-\varepsilon}$ unbdd

$\int_{c+\varepsilon}^{b}$ unbdd.

$$\underset{\text{unbdd}}{\int_{a+\varepsilon}^{b}} > \int_{c+\varepsilon}^{b}$$



$\Rightarrow$ Let $\phi(\varepsilon) = \int_{a+\varepsilon}^{b} f(x)\,dx$, $0 < \varepsilon \leq b-a$

$\int_a^b f(x)\,dx$ converges at $x = a$

if $\lim_{\varepsilon \to 0+} \int_{a+\varepsilon}^{b} f(x)\,dx$ exists finitely

Since $f(x) > 0$, $\forall x \in [a, b]$,

$\phi(\varepsilon) > 0$

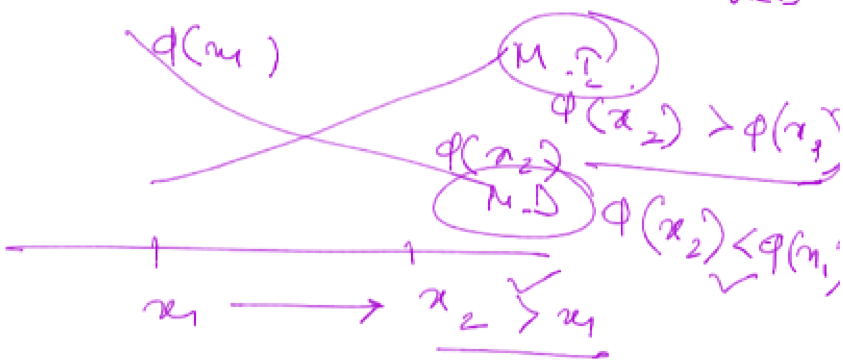
Let $\varepsilon_2 < \varepsilon_1$

$$\phi(\varepsilon_2) - \phi(\varepsilon_1) = \int_{a+\varepsilon_2}^{b} f(x)\,dx - \int_{a+\varepsilon_1}^{b} f(x)\,dx$$

$$= \int_{a+\varepsilon_2}^{a+\varepsilon_1} f(x)\,dx > 0$$

$\Rightarrow \phi(\varepsilon_2) > \phi(\varepsilon_1)$ when $\varepsilon_2 < \varepsilon_1$

$\therefore \phi(\varepsilon)$ is M.I. as ε decreases



$\lim_{\varepsilon \to 0+} \phi(\varepsilon)$ exists finitely iff

$\phi(\varepsilon)$ is bdd above.

So $\exists$ a (+)ve real k, s.t.

$\phi(\varepsilon) < k$, $0 < \varepsilon \leq b - a$.

$\Rightarrow \int_{a+\varepsilon}^{b} f(x)\,dx < k$, $0 < \varepsilon \leq b - a$.