Thm :— If f is bdd and int, then $|f|$ is bdd and int on $[a,b]$.

$\Rightarrow \quad P \quad , \quad M_r, m_r \longrightarrow f$

$\alpha, \beta \in [x_{r-1}, x_r] \quad M_r', m_r' \longrightarrow |f|$

$$\big||f(\alpha)| - |f(\beta)|\big| \le |f(\alpha) - f(\beta)|$$

$$\le M_r - m_r$$

$$\Rightarrow \quad M_r' - m_r' \le M_r - m_r$$

$$\Rightarrow \quad \sum_{r=1}^{n} (M_r' - m_r')\,\delta_r \le \sum_{r=1}^{n} (M_r - m_r)\,\delta_r$$

$$\omega(|f|, P) < \varepsilon$$

Converse may not true —

$$f(x) = \begin{cases} -1, & x \in \mathbb{Q}^c \cap [a,b] \\ +1, & x \in \mathbb{Q} \cap [a,b] \end{cases}$$

$$|f| = 1, \quad \forall x \in [a,b].$$

Thm :— If $f(x) \ge 0$, $\forall x \in [a,b]$

then $\int_a^b f(x)\,dx \ge 0$

$$\Rightarrow \quad m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

$$m \ge 0.$$

$$\int_a^b f(x)\,dx \ge 0.$$

Cor :- $f(x) \ge g(x)$, $\forall x \in [a,b]$

$$\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx.$$

$$\Rightarrow \quad F(x) = f(x) - g(x) \ge 0, \; \forall x \in [a,b].$$

$$\Rightarrow \int_a^b f(x)\,dx \ge 0$$

$$\Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx.$$

<u>Thm</u>:— If f is R-int on $[a,b]$ and $[c,d] \subseteq [a,b]$, then f is R-int on $[c,d]$.

$$\Rightarrow \quad P = \{ a = x_0 < x_1 < \dots < x_p = c < x_{p+1} < \dots < x_q = d < x_{q+1} < \dots < x_n = b \}$$

$\omega(f, P) < \varepsilon$ with $\|P\| < \delta$.

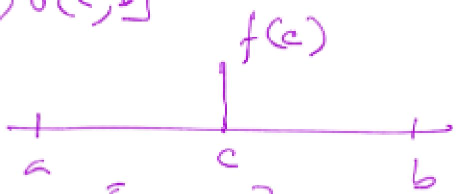
$$P_1 = \{ x_p = c < x_{p+1} < \dots < x_q = d \}$$

$\omega(f, P_1) < \varepsilon$, with $\|P_1\| < \delta$

<u>Thm</u>:— If f is a non-neg. cont. function on $[a,b]$ and if $f(c) > 0$ for $c \in [a,b]$, then $\int_a^b f(x)\,dx > 0$.

$$f(x) = \begin{cases} 0, & \forall x \in [a,c) \cup (c,b] \\ 1, & x = c \end{cases}$$

Not cont.

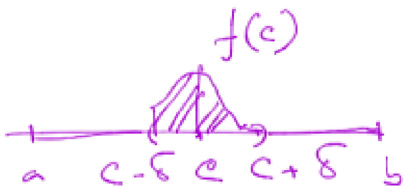


$$f_1(x) = 0, \quad \forall x \in [a,c]$$

$$f_2(x) = 0, \quad \forall x \in [c,b]$$

$$\int_a^b f(x)\,dx = 0.$$

$\Rightarrow \quad f(x) \ge 0, \ \forall x \in [a,b]$



$$\int^b f(x)\,dx > 0.$$

Now, f is cont. at $x = c$, $f(c) > 0$.

$$\therefore \quad f(x) > 0, \quad \forall x \in (c-\delta, c+\delta), \quad \delta > 0.$$

$$|f(x) - f(c)| < \varepsilon, \quad \forall x \in (c-\delta, c+\delta)$$

$$\Rightarrow \quad f(c) - \varepsilon < f(x) < f(c) + \varepsilon$$

$$\varepsilon = \tfrac{1}{2} f(c) > 0,$$

$$f(c) - \tfrac{1}{2} f(c) < f(x) < f(c) + \tfrac{1}{2} f(c)$$

$$\Rightarrow \quad \tfrac{1}{2} f(c) < f(x) < 3/2\, f(c)$$

$$\int_a^b f(x)\,dx = \underline{\int_a^{c-\delta} f(x)\,dx} + \int_{c-\delta}^{c+\delta} f(x)\,dx + \underline{\int_{c+\delta}^{b} f(x)\,dx}$$

$$\geq \int_{c-\delta}^{c+\delta} f(x)\,dx$$

$$> \tfrac{1}{2} f(c) \int_{c-\delta}^{c+\delta} dx$$

$$= \delta f(c) > 0.$$