Theorem:—

If f_1 and f_2 bounded and int. on $[a,b]$, then

i) (f_1+f_2) is also bdd and int. on $[a,b]$

and $\int_a^b (f_1 \pm f_2)(x)\,dx = \int_a^b f_1(x)\,dx \pm \int_a^b f_2(x)\,dx$

ii) $f_1 \cdot f_2$ is also bdd and int. on $[a,b]$.

iii) f_1/f_2 is bdd and int on $[a,b]$

if inf. of $|f_2|$ is +ve. ($f_2 \neq 0$ $\forall x$)

$\Rightarrow$ Let $K_1, K_2 > 0$ s.t.

$|f_1(x)| < K_1$, $|f_2(x)| < K_2$, $\forall x \in [a,b]$.

Let $K = \max\{K_1, K_2\}$

$\therefore |f_1(x)| < K$, $|f_2(x)| < K$

$\Rightarrow |f_1(x) + f_2(x)| < 2K$, $\forall x \in [a,b]$.

$\therefore f_1 + f_2$ is bdd on $[a,b]$.

Let $P = \{a = x_0 < x_1 < \dots < x_n = b\} \in \mathcal{P}[a,b]$

with $\|P\| < \delta$, $\delta > 0$.

For given $\varepsilon > 0$, $\exists\ \xi_r \in [x_{r-1}, x_r]$, $\forall r$

s.t. $\left| \sum_{r=1}^{n} f_1(\xi_r)\,\delta_r - \int_a^b f_1(x)\,dx \right| < \varepsilon/2$

and $\left| \sum_{r=1}^{n} f_2(\xi_r)\,\delta_r - \int_a^b f_2(x)\,dx \right| < \varepsilon/2$

with $\|P\| < \delta$.

$$\left| \sum_{r=1}^{n} (f_1+f_2)(\xi_r)\,\delta_r - \int_a^b f_1(x)\,dx - \int_a^b f_2(x)\,dx \right|$$

$$= \left| \sum_{r=1}^{n} f_1(\xi_r)\,\delta_r + \sum_{r=1}^{n} f_2(\xi_r)\,\delta_r \right.$$

$$\left. - \int_a^b f_1(x)\,dx - \int_a^b f_2(x)\,dx \right|$$

$$- \int_a^b f_1(x)\,dx - \int_a^b f_2(x)\,dx \Big|$$

$$\leq \left| \sum_{r=1}^{n} f_1(\xi_r)\delta_r - \int_a^b f_1(x)\,dx \right|$$

$$+ \left| \sum_{r=1}^{n} f_2(\xi_r)\delta_r - \int_a^b f_2(x)\,dx \right|$$

$$< \varepsilon/2 + \varepsilon/2 \quad \text{with } \|P\| < \delta$$

$$= \varepsilon$$

Hence $f_1 + f_2$ is int. on $[a,b]$ and

$$\int_a^b (f_1 + f_2)(x)\,dx = \int_a^b f_1(x)\,dx + \int_a^b f_2(x)\,dx.$$

ii) $f_1 \cdot f_2$ bdd (H/W)

$|f_1| < K$, $|f_2| < K$ $\forall x$.

$P = \{ a = x_0 < x_1 < \dots < x_n = b \}$

$$\left\{ \begin{array}{l} M_r', m_r' \rightarrow f_1 \\ M_r'', m_r'' \rightarrow f_2 \\ M_r, m_r \rightarrow f_1 \cdot f_2 = F \end{array} \right. \quad [x_{r-1}, x_r]$$

Let ξ, η be any two points of $[x_{r-1}, x_r]$

$$|F(\xi) - F(\eta)| = |f_1(\xi) f_2(\xi) - f_1(\eta) f_2(\eta)|$$

$$= |f_1(\xi)(f_2(\xi) - f_2(\eta)) + f_2(\eta)(f_1(\xi) - f_1(\eta))|$$

$$\leq |f_1(\xi)|\,|f_2(\xi) - f_2(\eta)| + |f_2(\eta)|\,|f_1(\xi) - f_1(\eta)|$$

$$(M_r - m_r) < K(M_r'' - m_r'') + K(M_r' - m_r') \quad Ⓐ$$

$(\because \xi, \eta$ are arbitrary$)$

Let $\delta > 0$ be s.t. $\|P\| < \delta$.

f_1, f_2 int $\Rightarrow$ o.s. of f_1, f_2 can be made less than $\frac{\varepsilon}{2K}$ with $\|P\| < \delta$.

from (A)

$$\sum_{r=1}^{n} (M_r - m_r)\,\delta_r \le K\left[\sum_{r=1}^{n} (M_r'' - m_r'')\,\delta_r + \sum_{r=1}^{n} (M_r' - m_r')\,\delta_r\right]$$

$$< K \cdot 2\varepsilon/2K = \varepsilon$$

$\omega(f_1 \cdot f_2, P) < \varepsilon$, with $\|P\| < \delta$.

$\therefore$ $f_1 \cdot f_2$ is int. on $[a, b]$.

iii) $|f_2(x)| \ge \mu > 0$, $\mu = \inf\limits_{x \in [a,b]} f_2(x)$

bdd $\rightarrow$ $|f_1(x)| < K$, $\forall x$

$$\left|\frac{f_1(x)}{f_2(x)}\right| < \frac{K}{\mu}, \forall x.$$

$$|F(\xi) - F(\eta)| = \left|\frac{f_1(\xi)}{f_2(\xi)} - \frac{f_1(\eta)}{f_2(\eta)}\right|$$

$$= \frac{1}{|f_2(\xi) f_2(\eta)|}\,|f_1(\xi) f_2(\eta) - f_1(\eta) f_2(\xi)|$$

$$\le \frac{1}{\mu^2} \,|\qquad\qquad|$$