Thm:— If $|f(x)| \le K$, $\forall x \in [a,b]$. If f is int. on $[a,b]$, then $\left|\int_a^b f(x)\,dx\right| \le K(b-a)$

$\Rightarrow$ $-K \le m \le f(x) \le M \le K$

$$-K(b-a) \le m(b-a) \le \int_a^b f(x)\,dx \le M(b-a) \le K(b-a)$$

$\#$ f R-int $\Rightarrow$ $|f|$ R-int.

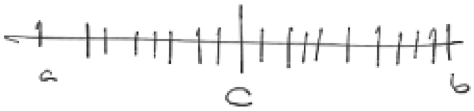
converse is not true.

$$f = \begin{cases} 1, & x \in \mathbb{Q} \cap [a,b] \\ -1, & x \in \mathbb{Q}^c \cap [a,b] \end{cases}$$ is not int.

$|f| = 1$, $\forall x \in [a,b]$ is int.

Thm:— If f is bounded and integrable on $[a,b]$ and c be any point of $[a,b]$. Then f is int. on $[a,c]$ and $[c,b]$. conversely if f is int on $[a,c]$ and $[c,b]$, then f is int on $[a,b]$.

Moreover, $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$



$\Rightarrow$ Let f is int. on $[a,b]$. so for given $\varepsilon > 0$, $\exists\, \delta > 0$ s.t.

$$\omega(f,P) < \varepsilon, \quad \forall P \text{ with } \|P\| < \delta$$

Let $P = \{a = x_0 < x_1 < \dots < x_k = c < x_{k+1} < x_{k+2} < \dots < x_n = b\}$

$$\sum_{r=0}^{k} \delta_r + \sum_{r=k+1}^{n} \delta_r = (c-a) + (b-c) = b-a.$$

Let us take, $P_1 = \{a = x_0 < x_1 < \dots < x_k = c\}$
and $P_2 = \{c = x_{k+1} < x_{k+2} < \dots < x_n = b\}$

be two partitions on $[a, c]$ and $[c, b]$ resp.

Also, $P = P_1 \cup P_2$.

$M_r =$

$m_r =$

$$U(f, P_1) = \sum_{r=1}^{k} M_r \delta_r, \quad L(f, P_1) = \sum_{r=1}^{k} m_r \delta_r$$

$$U(f, P_2) = \sum_{r=k+2}^{n} M_r \delta_r, \quad L(f, P_2) = \sum_{r=k+2}^{n} m_r \delta_r$$

$$U(f, P) = U(f, P_1) + U(f, P_2)$$

$$L(f, P) = L(f, P_1) + L(f, P_2)$$

$$\omega(f, P) = U(f, P) - L(f, P)$$

$$= \{U(f, P_1) + U(f, P_2)\} - \{L(f, P_1) + L(f, P_2)\}$$

$$= \{U(f, P_1) - L(f, P_1)\} + \{U(f, P_2) - L(f, P_2)\}$$

$$= \omega(f, P_1) + \omega(f, P_2) \quad \text{—— (A)}$$

We have $\omega(f, P) < \varepsilon, \ \forall P$ with $\|P\| < \delta$.

$\Rightarrow 0 \leq \omega(f, P_1) + \omega(f, P_2) < \varepsilon, \ \forall P_1, P_2 \in \mathcal{P}[a,b]$ with $\|P_1\| < \delta$, $\|P_2\| < \delta$

$\therefore \omega(f, P_1) < \varepsilon, \ \forall P_1$ with $\|P_1\| < \delta$

$\omega(f, P_2) < \varepsilon, \ \forall P_2$ with $\|P_2\| < \delta$.

$\therefore f$ is int on $[a, c]$ and $[c, b]$ resp.

Conversely, let f is int on $[a, c]$ & $[c, b]$.

So $\exists \ Q_1, Q_2 \in \mathcal{P}[a, b]$ s.t.

$\omega(f, Q_1) < \varepsilon/2, \ \forall Q_1$ with $\|Q_1\| < \delta$

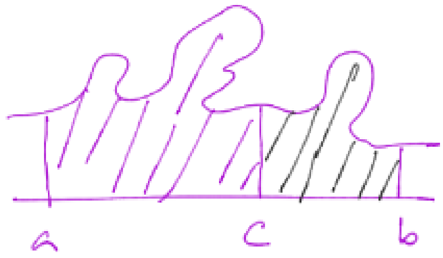
$\omega(f, Q_2) < \varepsilon/2, \ \forall Q_2$ with $\|Q_2\| < \delta$

Let us take $Q = Q_1 \cup Q_2 \in \mathcal{P}[a, b]$

and $\|Q\| < \delta$.

$\omega(f, Q) = \omega(f, Q_1) + \omega(f, Q_2)$

$$\omega(f, y) = \omega(f, y_1) + \omega(f, y_2)$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon, \ \forall P, \ \|P\| < \delta.$$
$\therefore$ f is int on $[a, b]$.



\# $U(f, P_1) \geq \overline{\int_a^c} f(x)\,dx$

$U(f, P_2) \geq \overline{\int_c^b} f(x)\,dx$

$U(f, P) = U(f, P_1) + U(f, P_2)$

$\Rightarrow U(f, P) \geq \overline{\int_a^c} f(x)\,dx + \overline{\int_c^b} f(x)\,dx, \ \forall P$

$\Rightarrow \overline{\int_a^b} f(x)\,dx \geq \overline{\int_a^c} f(x)\,dx + \overline{\int_c^b} f(x)\,dx \longrightarrow (1)$

Now, f is int over $[a, b]$, $[a, c]$ & $[c, b]$

So from (1), $\int_a^b f(x)\,dx \geq \int_a^c f(x)\,dx + \int_c^b f(x)\,dx. \longrightarrow (2)$

$L(f, P) = L(f, P_1) + L(f, P_2)$

$L(f, P) \leq \underline{\int_a^c} f(x)\,dx + \underline{\int_c^b} f(x)\,dx, \ \forall P$

$\Rightarrow \underline{\int_a^b} f(x)\,dx \leq \underline{\int_a^c} f(x)\,dx + \underline{\int_c^b} f(x)\,dx \longrightarrow (3)$

$\Rightarrow \int_a^b f(x)\,dx \leq \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \longrightarrow (4)$

from (2), (4),

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

<u>Alternative</u> Proof:—

$$\left| \sum_{r=1}^{n} f(\xi_r)\,\delta_r - \int_a^b f(x)\,dx \right| < \varepsilon/3$$

$$\left| \sum_{r=1}^{k} f(\xi_r)\,\delta_r - \int_a^c f(x)\,dx \right| < \varepsilon/3$$

$$\left| \sum_{r=k+2}^{n} f(\xi_r)\,\delta_r - \int_c^b f(x)\,dx \right| < \varepsilon/3$$

$$0 \leq \left| \int_a^b f(x)\,dx - \int_a^c f(x)\,dx - \int_c^b f(x)\,dx \right|$$

$$= \left| \int_a^b f(x)\,dx - \sum_{r=1}^{n} f(\xi_r)\,\delta_r + \sum_{r=1}^{k} f(\xi_r)\,\delta_r + \sum_{r=k+2}^{n} f(\xi_r)\,\delta_r - \int_a^c f(x)\,dx - \int_c^b f(x)\,dx \right|$$

$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon .$$

$$\therefore \int_a^b f(x)\,dx = \int_c^c f(x)\,dx + \int_c^b f(x)\,dx .$$