Theorem :—

If $f:[a,b]\to\mathbb{R}$ be bounded on $[a,b]$ and c be any nonzero constant, then

i) $\overline{\int_a^b} c f(x)\,dx = c\overline{\int_a^b} f(x)\,dx$

and $\underline{\int_a^b} c f(x)\,dx = c\underline{\int_a^b} f(x)\,dx$ if $c>0$

ii) $\overline{\int_a^b} c f(x)\,dx = c\underline{\int_a^b} f(x)\,dx$

and $\underline{\int_a^b} c f(x)\,dx = c\overline{\int_a^b} f(x)\,dx$, if $c<0$.

Also if f is bdd and integrable on $[a,b]$ then cf is also bdd and integrable on $[a,b]$ and $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, c is any real no.

$\Rightarrow$ i) If $c>0$.

$$\overline{\int_a^b} cf(x)\,dx = \inf_{P\in\mathcal{P}[a,b]} U(cf,P)$$

$$= \inf_{P\in\mathcal{P}[a,b]} c\,U(f,P)$$

$$U(cf,P)=\sum cM_r\delta_r = c\sum M_r\delta_r = c\,U(f,P)$$

$$= c\inf_{P\in\mathcal{P}[a,b]} U(f,P)$$

$$= c\overline{\int_a^b} f(x)\,dx \quad \text{—} \ (1)$$

Ily, $\underline{\int_a^b} cf(x)\,dx = c\underline{\int_a^b} f(x)\,dx$, $c>0$. — (2)

ii) If $c<0$. let $c=-d$, $d>0$

$$\overline{\int_a^b} cf(x)\,dx = \overline{\int_a^b} -d f(x)\,dx$$

$$= d\overline{\int_a^b} -f(x)\,dx$$

$$= d\inf_{P\in\mathcal{P}[a,b]} U(-f,P)$$

$$U(-f,P) = \sum N_r\delta_r = -\sum m_r\delta_r$$

$$= c \inf_{P \in \mathcal{P}[a,b]} -L(f,P)$$

$$= -c \sup_{P \in \mathcal{P}[a,b]} L(f,P)$$

$$= c \underline{\int_a^b} f(x)\,dx \quad \text{— (3)}$$

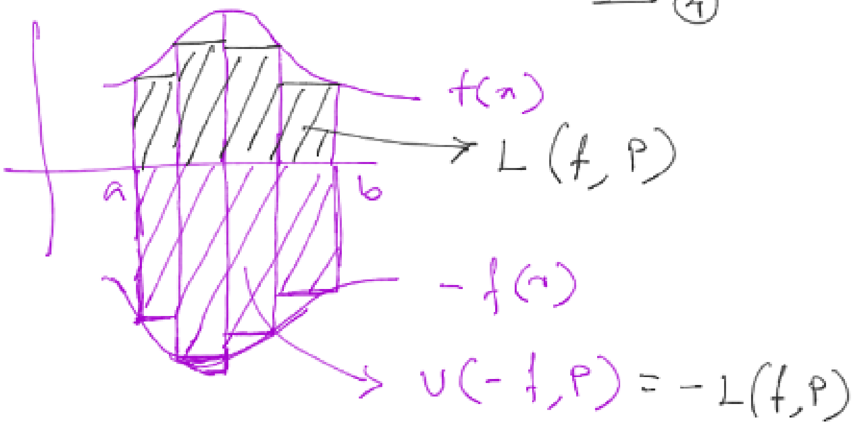
$\left\{ \begin{array}{l} f \to 1, 2, \dots, 10 \\ -f \to -1, -2, \dots, -10 \end{array} \right.$

$\sup -f \to -1$

$= -\inf f$

$\inf(-A) = -\sup A$

||ly, $\underline{\int_a^b} c f(x)\,dx = c \overline{\int_a^b} f(x)\,dx, \quad c < 0. \quad$ — (4)



\# f is bounded. Then $\exists\, K > 0$ s.t.

$c \in \mathbb{R}$, $|f(x)| \le K, \ \forall x.$

$$|c f(x)| = |c|\,|f(x)| \le |c| . K, \ \forall x \in [a,b]$$

$\therefore$ cf is bounded on $[a,b]$.

f is int. on $[a,b]$, so $\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$

$= \int_a^b f(x)\,dx$ — (5)

From (1), (2), (3), (4), (5)

$$\overline{\int_a^b} c f(x)\,dx = \underline{\int_a^b} c f(x)\,dx = c \int_a^b f(x)\,dx$$

$$\Rightarrow \int_a^b c f(x)\,dx = c \int_a^b f(x)\,dx$$

cf is integrable on $[a,b]$.

Theorem :— If $f : [a,b] \to \mathbb{R}$ be bounded f^n and integrable on $[a,b]$ and M, m

be the sup and inf of f on $[a, b]$, then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a), \quad b \ge a$$

$$m(b-a) \ge \int_a^b f(x)\,dx \ge M(b-a), \quad b \le a.$$

$\Rightarrow$ $\begin{array}{l} m_r = \inf f \\ M_r = \sup f \\ \quad x \in [x_{r-1}, x_r] \end{array}$ $\qquad P = \{a = x_0 < x_1 < \dots \dots x_n = b\}$

$$m \le m_r \le M_r \le M$$

$$\Rightarrow \sum_{r=1}^{n} m\,\delta_r \le \sum_{r=1}^{n} m_r\,\delta_r \le \sum_{r=1}^{n} M_r\,\delta_r \le \sum_{r=1}^{n} M\,\delta_r$$

$$\Rightarrow m(b-a) \le L(f, P) \le U(f, P) \le M(b-a)$$

$$\Rightarrow m(b-a) \le L(f, P) \le L \le U \le U(f, P) \le M(b-a)$$

$$\Rightarrow m(b-a) \le L = U \le M(b-a)$$

$$\Rightarrow m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$

$P \in \mathcal{P}[a, b]$, $\xi_r \in [x_{r-1}, x_r]$

$$m \le m_r \le f(\xi_r) \le M_r \le M$$

$$\Rightarrow \sum m\,\delta_r \le \sum m_r\,\delta_r \le \sum_{r=1}^{n} f(\xi_r)\,\delta_r$$

$$\le \sum M_r\,\delta_r \le \sum M\,\delta_r$$

$$\Rightarrow m(b-a) \le \sum f(\xi_r)\,\delta_r \le M(b-a)$$

$$\Rightarrow m(b-a) \le \lim_{\|P\| \to 0} \sum f(\xi_r)\,\delta_r \le M(b-a)$$

$$\Rightarrow m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

Thm:— f is bdd and int. on $[a, b]$. Then $\exists$ a real no μ s.t. $m \le \mu \le M$

and $\int_a^b f(x)\,dx = \mu(b-a)$.

$\Rightarrow \quad m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$

Let $\mu = \frac{1}{(b-a)} \int_a^b f(x)\,dx$.

$\therefore \quad m \leq \mu \leq M$

and $\int_a^b f(x)\,dx = \mu(b-a)$.

Thm: If f is cont. on $[a,b]$, then $\exists\ \xi$, $a \leq \xi \leq b$, s.t.

$$\int_a^b f(x)\,dx = (b-a) f(\xi) = h\, f(a+\theta h), \quad 0 < \theta < 1.$$

$\Rightarrow$ As f is cont., f attains every pt in between a and b, let $f(\xi) = \mu$, $a \leq \xi \leq b$.