Riemann

Problem:—

1) $f : [0,1] \to \mathbb{R}$

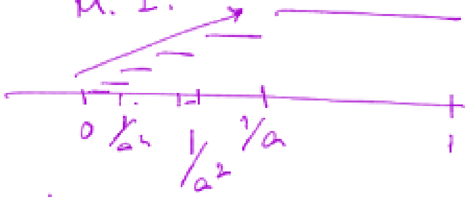
$$f(x) = \frac{1}{a^{r-1}}, \quad \frac{1}{a^r} < x \le \frac{1}{a^{r-1}}; \quad r = 1, 2, \ldots$$

$a > 2$.

Is f R-int on $[0,1]$? Find $\int_0^1 f(x)\,dx$.

$\Rightarrow$ f has infinite no of pt of discontinuities $\{0, \frac{1}{a}, \frac{1}{a^2}, \frac{1}{a^3}, \ldots\}$ having only one limit pt 0. (finite).

M. I. $\Rightarrow$ R-int



$\therefore$ f is R-int.

$$f(x) = \begin{cases} 1, & \frac{1}{a} < x \le 1 \\ 1/a, & 1/a^2 < x \le 1/a \\ 1/a^2, & 1/a^3 < x \le 1/a^2 \\ \vdots \end{cases}$$

$$\int_{1/a^n}^{1} f(x)\,dx = \int_{1/a}^{1} f(x)dx + \int_{1/a^2}^{1/a} + \cdots + \int_{1/a^n}^{1/a^{n-1}} f(x)\,dx$$

$$= \int_{1/a}^{1} 1 \cdot dx + \int_{1/a^2}^{1/a} \frac{1}{a}\,dx + \cdots + \int_{1/a^n}^{1/a^{n-1}} \frac{1}{a^{n-1}}\,dx$$

$$= \left(1 - \frac{1}{a}\right) + \frac{1}{a}\left(\frac{1}{a} - \frac{1}{a^2}\right) + \cdots + \frac{1}{a^{n-1}}\left(\frac{1}{a^{n-1}} - \frac{1}{a^n}\right)$$

$$= \frac{a-1}{a} + \frac{a-1}{a^3} + \cdots + \frac{a-1}{a^{2n-1}}$$

$$= \frac{a-1}{a}\left[1 + \frac{1}{a^2} + \cdots + \frac{1}{a^{2n-2}}\right]$$

$$= \left(\frac{a-1}{a}\right) \frac{1 - \left(\frac{1}{a^2}\right)^n}{1 - \frac{1}{\cdot}}$$

$$\int_0^1 f(x)\,dx = \underset{n\to\infty}{Lt} \int_{1/a^n}^{1} f(x)\,dx = \frac{a-1}{a}\cdot\frac{1}{1-\frac{1}{a^2}}$$

as $a > 2$

$$= \frac{a-1}{a}\cdot\frac{a^2}{a^2-1} = \frac{a}{a+1}.$$

2) Let $f(x) = 0$ for $x = \frac{n}{n+1}, \frac{n+1}{n},$ $n = 1, 2, 3, \dots$

$= 1$, elsewhere

$f : [0, 2] \to \mathbb{R}$

find $\overline{\int_0^2} f(x)\,dx$. Is f R-int on $[0,2]$? Is f cont. at $x = 1$?

$\Rightarrow$ $f(x) = 0$ for $x = \frac{1}{2}, 2, \frac{2}{3}, \frac{3}{2}, \frac{3}{4}, \frac{4}{3}$ ......

f has infinite no of points of discontinuities having only one limit point 1.

$\therefore$ f is R-int on $[0,2]$.

$$\overline{\int_0^2} f(x)\,dx = \inf_n \left\{ \sum_{r=1}^{n} M_r \delta_r \right\}$$

$$= \inf \sum_{r=1}^{n} 1 \cdot \delta_r$$

$U(f,P) = 2$ for any P

$$= \inf 2 = 2$$

$$\therefore \overline{\int_0^2} f(x)\,dx = \int_0^2 f(x)\,dx = 2.$$

$$\lim_{x\to 1-} f(x) = 0, \quad \lim_{x\to 1+} f(x) = 0$$

$f(1) = 1.$

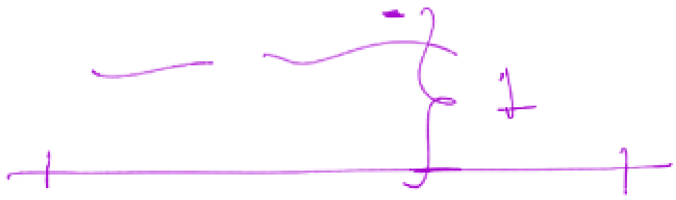
$\therefore$ f is not cont. at $x = 1$.

3) $f : [0,1] \to \mathbb{R}$

$$f(x) = 0, \quad x \in \mathbb{Q}^c \cap [0,1]$$
$$= 1/q, \quad x = p/q,$$ where p & q are (+)ve integers prime to each other.

Show that f is R-int on $[0,1]$ and $\int_0^1 f(x)\,dx = 0$.

$\Rightarrow$



$q = 1200$

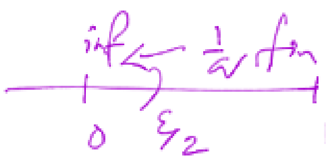
$\frac{1}{q}$

$\frac{p}{q} = \frac{1}{1200}, \frac{2}{1200}, \frac{7}{1200}, \ldots \frac{1199}{1200}$

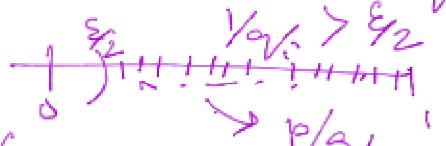
$\Rightarrow$ f is bounded on $[0,1]$ and the oscillation of f is atmost 1.

Let $q > 0$ be given, then $\exists$ only finite no of (+)ve integers q's, s.t. $\frac{1}{q} > \varepsilon/2$ for given $\varepsilon > 0$.

$\exists$ some finite no of



values of $q \to q_1, q_2, \ldots, q_m$. So $\exists$ finite no of points p/q_i, $i = 1(1)m$; where $\frac{1}{q_i} > \varepsilon/2$



We enclose these points p/q_i ...

by non-overlapping closed intervals
$[a_1, b_1], [a_2, b_2], \ldots, [a_m, b_m]$ of $[0,1]$.

s.t. $p/q_i \in [a_i, b_i]$ for $i = 1(1)m$.

and moreover $\sum_{i=1}^{m} (b_i - a_i) < \varepsilon/2$

$$\left[\text{o. s.} \longrightarrow \sum_{i=1}^{m} 1.(b_i - a_i) \right]$$

at each point p/q in the remaining $(m+1)$ subintervals $[0, a_1], [b_1, a_2], \ldots, [b_m, 1]$

$1/q < \varepsilon/2$.

oscillation of $f = | f(x) - f(p/q) |$

$$= \frac{1}{q} < \varepsilon/2$$

$P = \{ 0 < a_1 < b_1 < a_2 < b_2 \cdots < a_m < b_m < 1 \}$ of $[0, 1]$.

$$\omega(f, P) = \sum_{i=1}^{m} (M_i - m_i)(b_i - a_i) + \sum_{r=0}^{m} (M_r - m_r)(a_{r+1} - b_r)$$

$$< \sum_{i=1}^{m} 1 (b_i - a_i) + \sum_{r=0}^{m} \frac{1}{q} (a_{r+1} - b_r)$$

$$< \varepsilon/2 + \varepsilon/2 \sum (a_{r+1} - b_r)$$

$$< \varepsilon/2 + \varepsilon/2 \cdot 1$$

$$= \varepsilon.$$

$$\int_{0}^{1} f(x)\, dx = 0 = \int_{0}^{1} f(x)\, dx.$$