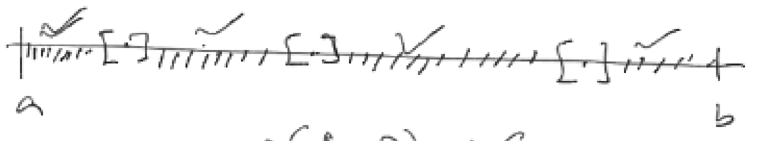


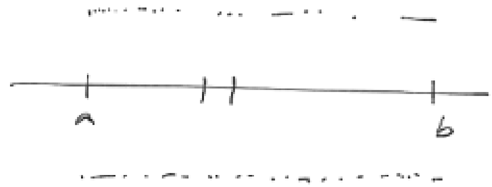
$P \quad \textcircled{\varepsilon}$

$\omega(f, P) < \varepsilon$

<u>Thm</u> :— If a bounded f^n f on $[a, b]$ has a set of points of discontinuities having finite no of limit points on $[a, b]$, then f is R-int. on $[a, b]$.

$$f(x) = \begin{cases} -1, & x \in \mathbb{Q}^c \cap [a,b] \\ 1, & x \in \mathbb{Q} \cap [a, b] \end{cases}$$

$\mathbb{R} \cap [a, b]$



$M_r = 1, \quad m_r = -1$

$\omega(f, P) = \sum (1+1)\delta_r = 2(b-a) > \varepsilon$

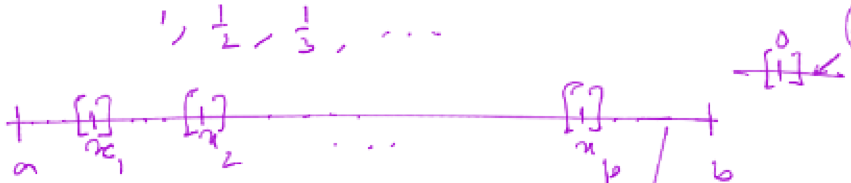
$U(f, P) = 1.(b-a), \quad L(f, P) = -1(b-a)$

$U = b - a \qquad L = -(b-a)$

$U \neq L$

<u>Proof</u> :— <u>H/W</u>

$1, \frac{1}{2}, \frac{1}{3}, \ldots$



$\frac{1}{n} \rightarrow$ finite

$1, \frac{1}{2}, \ldots, \frac{1}{N}$

limit p^t

finite no of p^t of discont.
Hence R-int.

<u>Thm</u> :— If f and g be two bounded f^n on $[a, b]$ s.t. $f(x) = g(x)$ except for finitely many points on $[a, b]$. If $g(x)$ is R-int. over $[a, b]$, then f is also R-int. on $[a, b]$. Moreover, $\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$.

<u>H/W</u>

Proof :—

Problem: $f : [0,4] \to \mathbb{R}$ is defined by $f(x) = x[x]$, $\forall x \in [0,4]$. Is f R-int on $[0,4]$? If so, evaluate $\int_0^4 f(x)\,dx$.

$\Rightarrow$
$$f(x) = \begin{cases} 0, & 0 \le x < 1 \\ x, & 1 \le x < 2 \\ 2x, & 2 \le x < 3 \\ 3x, & 3 \le x < 4 \\ 4x, & x = 4 \end{cases}$$

Points of discontinuities $x = 1, 2, 3, 4$ (finite)
Hence f is R-int.

Let us define g_1, g_2, g_3, g_4 on $[0,1]$, $[1,2]$, $[2,3]$, $[3,4]$ respectively as

$$g_1(x) = 0, \quad 0 \le x \le 1$$
$$g_2(x) = x, \quad 1 \le x \le 2$$
$$g_3(x) = 2x, \quad 2 \le x \le 3$$
$$g_4(x) = 3x, \quad 3 \le x \le 4$$

Here $f(x) = g_1(x)$, $\forall x \in [0,1]$ except at $x = 1$.

$$\int_0^1 f(x)\,dx = \int_0^1 g_1(x)\,dx = \int_0^1 0\,dx = 0.$$

$$\int_1^2 f(x)\,dx = \int_1^2 g_2(x)\,dx = \int_1^2 x\,dx = \frac{x^2}{2}\Big|_1^2 = 3/2$$

$$\int_2^3 f(x)\,dx = \int_2^3 g_3(x)\,dx = \int_2^3 2x\,dx$$

$$\int_3^4 f(x)\,dx = \int_3^4 g_4(x)\,dx = \int_3^4 3x\,dx$$

$$\int_0^4 f(x)\,dx = \int_0^1 + \int_1^2 + \int_2^3 + \int_3^4.$$

2) $f : [0,1] \to \mathbb{R}$

$$f(x) = (-1)^{r-1}, \quad \frac{1}{r+1} < x \le \frac{1}{r}; \; r = 1, 2, 3, \ldots$$

$f(0) = 0$

Is f R-int? If so evaluate $\int_0^1 f(x)\,dx$.

$0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$

limit pt $\longrightarrow \{0\}$ finite

$$\int_{\frac{1}{n}}^{1} f(x)\,dx = \int_{\frac{1}{2}}^{1} f(x)\,dx + \int_{1/3}^{1/2} f(x)\,dx + \dots + \int_{1/n}^{1/(n-1)} f(x)\,dx$$

$g_1, g_2, \dots, g_{n-1}$

$g_1(x) = 1, \ \frac{1}{2} \le x \le 1, \quad g_2(x) = -1, \ \frac{1}{3} \le x \le \frac{1}{2}$

$\dots \quad g_{n-1}(x) = (-1)^{n-2}, \ \frac{1}{n} \le x \le \frac{1}{n-1}$

$$\int_{1/n}^{1} f(x)\,dx = \int_{1/2}^{1} 1\,dx + \int_{1/3}^{1/2} (-1)\,dx + \dots + \int_{1/n}^{1/(n-1)} (-1)^{n-2}\,dx$$

$$= \left(1 - \frac{1}{2}\right) - \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) - \dots + (-1)^{n-2}\left(\frac{1}{n-1} - \frac{1}{n}\right)$$

$$= \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} - \dots + (-1)^{n-2}\frac{1}{n-1}\right) + \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots - (-1)^{n-2}\frac{1}{n}\right) - 1$$

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty} \int_{1/n}^{1} f(x)\,dx$$

$$= \log(1+1) + \log(1+1) - 1$$

$$= 2\log 2 - 1 = \log 4 - 1 = \log\frac{4}{e}.$$

$$\log(1+x) = 1 - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \dots$$