$\omega(f, P) = U(f, P) - L(f, P)$ $\quad x_n$ M.D.
≥ 0 $\quad \{x_{P}\} < \varepsilon$

$\omega(f, P)$ decreases as we refine P. $\quad x_n < \varepsilon, \ \forall n \geq P$

$\omega(f, P) < \varepsilon$ $\quad \|P\| < \delta_1$

$\omega(f, P_1) < \varepsilon'$, $\quad P \subseteq P_1$

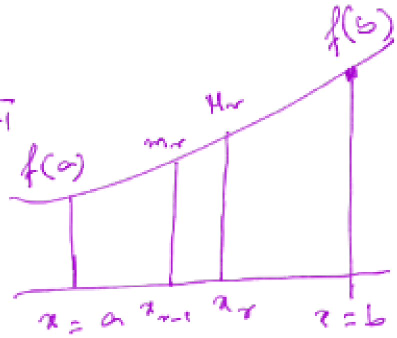
$\omega(f, Q) < \varepsilon$, $\forall Q$ with $\|Q\| < \delta_1$

Theorem:— Let f be monotone on $[a, b]$, then f is R-int on $[a, b]$.

$\Rightarrow$ Case I:— Let f be monotone increasing.

Let $P = \{a = x_0 < x_1 < \dots < x_n = b\}$ with $\|P\| < \delta$, where $\delta = \dfrac{\varepsilon}{f(b) - f(a) + 1}$

$M_r = f(x_r)$, $m_r = f(x_{r-1})$ $\forall r = 1(1)n$.



$$\omega(f, P) = \sum_{r=1}^{n} (M_r - m_r)\,\delta_r$$
$$< \delta \sum_{r=1}^{n} f(x_r) - f(x_{r-1})$$
$$= \delta\,(f(b) - f(a))$$
$$< \varepsilon \quad \text{with } \|P\| < \delta.$$

So, by N-S condition, f is R-int.

H/W. Case II:— when f is M.D.

Thm:— If f is cont. f^n on $[a, b]$, then f is R-int. on $[a, b]$.

$\Rightarrow$ Here f is cont. on bounded and closed interval $[a, b]$

so, f is unif. cont.

For given $\varepsilon > 0$, $\exists\ \delta(\varepsilon) > 0$ s.t.

$$|f(x_1) - f(x_2)| < \frac{\varepsilon}{b-a}, \quad \text{whenever } |x_1 - x_2| < \delta(\varepsilon)$$
$\quad$ (1) $\quad x_1, x_2 \in [a, b]$

Let $P \in \mathcal{P}[a, b]$ with $\|P\| < \delta$.

$P = \{a = x_0 < x_1 < \dots < x_n = b\}$.

$M_r = \sup\limits_{x \in [x_{r-1}, x_r]} f(x)$ $\qquad m_r = \inf\limits_{x \in [x_{r-1}, x_r]} f(x)$

Since f is cont. on $[a, b]$, $\exists\ \xi_r, \eta_r \in [x_{r-1}, x_r]$

s.t. $M_r = f(\xi_r)$, $\forall r = 1(1)n$

$m_r = f(\eta_r)$, $\forall r = 1(1)n$

$M_r - m_r = f(\xi_r) - f(\eta_r) < \varepsilon/(b-a)$

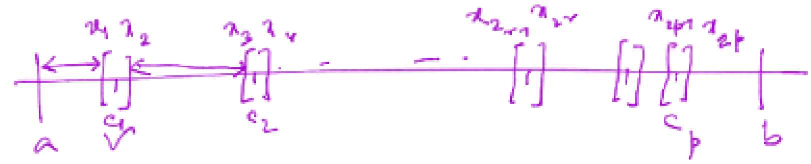
as $|\xi_r - \eta_r| \le |x_r - x_{r-1}| < \delta$.

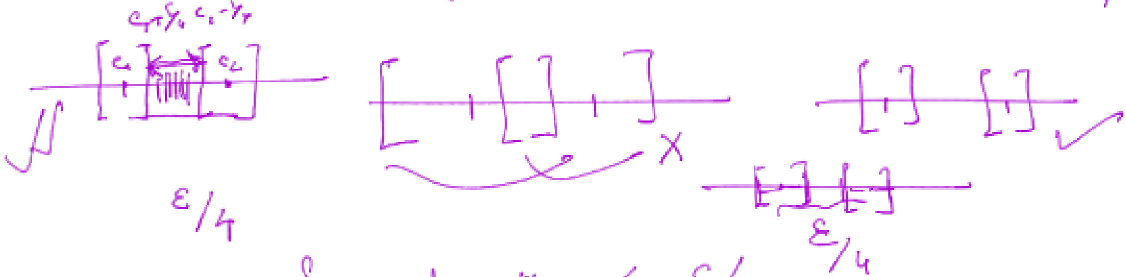
$$\omega(f, P) = \sum_{r=1}^{n} (M_r - m_r)\,\delta_r$$

$$< \frac{\varepsilon}{b-a}\cdot \sum_{r=1}^{n} \delta_r = \varepsilon, \text{ with } \|P\| < \delta.$$

Thm:— If a bdd f^n f on $[a,b]$ has a set of finite points of discontinuities on $[a,b]$ then f is R-int. on $[a,b]$.

$\Rightarrow$





Sum length $< \varepsilon/M-m$

$$\sum (M_r - m_r)\,\delta_r < (M-m)\sum \delta_r$$

$$< (M-m)\frac{\varepsilon}{(M-m)} = \varepsilon$$

———— x ————

Let f is discont. at p (finite) no of points $c_1, c_2, \dots, c_p$ in $[a,b]$.

Let us enclose these points of discontinuities by non overlapping closed intervals $[x_1, x_2], [x_3, x_4], \dots, [x_{2r-1}, x_{2r}], \dots [x_{2p-1}, x_{2p}]$ in such a way that $\sum_{r=1}^{p} (x_{2r} - x_{2r-1}) < \frac{\varepsilon}{2(M-m)}$ *

Now O.S. (oscillatory sum) arising from these subintervals is $\sum_{r=1}^{p} (M_r - m_r)(x_{2r} - x_{2r-1})$

where $M_r = \sup_{x \in [x_{2r-1}, x_{2r}]} f(x)$, $m_r = \inf_{x\in[\dots]} f(x)$

$\ldots [x_{2r-1}, x_{2r}] \qquad x \in [x_{2r-1}, x_{2r}]$

* Let M, m be the sup, inf of f in $[a,b]$.

O.S. in these subintervals is

$$\sum_{r=1}^{p} (M_r - m_r)(x_{2r} - x_{2r-1}) < (M-m) \sum_{r=1}^{p} (x_{2r} - x_{2r-1})$$

$$< (M-m) \frac{\varepsilon}{2(M-m)}$$

$$= \varepsilon/2 \longrightarrow (A)$$

In the remaining $(p+1)$ subintervals $[a, x_1], [x_2, x_3], \ldots, [x_{2p}, b]$, f is cont, hence integrable.

So we can take a partition P_i, $i = 1(1)p+1$ of these subint. separately in such a way that O.S. arising from these subintervals is less than $\varepsilon/2(p+1) \longrightarrow (B)$

Let $P = P_1 \cup P_2 \cup \cdots \cup P_{p+1} \bigcup_{r=1}^{p} [x_{2r-1}, x_{2r}]$

Then O.S. arising from $[a,b]$ w.r.t. the partition P i.e. $\omega(f, P) < \frac{\varepsilon}{2} + (p+1) \cdot \frac{\varepsilon}{2(p+1)}$

$$= \frac{\varepsilon}{2} + \varepsilon/2 \qquad [\text{From } (A) \,\&\, (B)]$$

$$= \varepsilon.$$

Hence f is R-int.