Thm :— If M and m be the sup and inf of a bdd f^n f on $[a, b]$, then $(M-m)$, the oscillation of f on $[a, b]$ is the sup of $S = \{ |f(\alpha) - f(\beta)| : \alpha, \beta \in [a, b] \}$

$\Rightarrow$ $M - m$ $\qquad$ $\underline{|f(\alpha) - f(\beta)| \leq M - m}$

$M = \sup f$ $\qquad$ $m = \inf f$.

$|f(\alpha)| \leq M, \forall \alpha$ $\quad$, $\quad$ $|f(\beta)| \geq m, \forall \beta$

$$f(\alpha) - f(\beta) \leq M - m \qquad -M \leq f(\alpha) \leq M$$
$$f(\alpha) - f(\beta) \geq m - M = -(M-m) \qquad m \leq -f(\beta) \leq -m$$

$$|f(\alpha) - f(\beta)| \leq M - m \quad \text{— (A)}$$

Again, $\quad f(\xi) \geq M - \varepsilon/2$, for $\xi \in [a, b]$

$\quad f(\eta) \leq m + \varepsilon/2$, for $\eta \in [a, b]$

$$f(\xi) - f(\eta) \geq (M - m) - \varepsilon/2 - \varepsilon/2$$

$$f(\xi) - f(\eta) \geq (M - m) - \varepsilon$$

— (B) for $\xi, \eta \in [a, b]$

$(M - m)$ is the sup of $\{ |f(\alpha) - f(\beta)| : \alpha, \beta \in [a, b] \}$

<u>Integral function of</u> f :—

Let f be R-int on $[a, b]$.

A function $\phi(t)$, $a < t \leq b$, defined by $\phi(t) = \int_a^t f(x)\, dx$, with $\phi(a) = 0$ is called integral f^n of f.

* $\phi(t) = \int_t^b f(x)\,dx, \quad a \le t \le b.$

Thm :— If f be integrable f^n on $[a,b]$ then the integral f^n $\phi(t) = \int_a^t f(x)\,dx, \quad a < t \le b$ is cont. on $[a,b]$.

$\Rightarrow$ f is bdd, $|f(x)| \le K$, $\forall x$, $K > 0$

Let $c \in [a,b]$ and $h > 0$ be such that $c+h \in [a,b]$.

$$\phi(c) = \int_a^c f(t)\,dt$$
$$\phi(c+h) = \int_a^{c+h} f(t)\,dt$$

$$\begin{aligned}
|\phi(c+h) - \phi(c)| &= \left| \int_c^{c+h} f(t)\,dt \right| \\
&\le \left| \int_c^{c+h} |f(t)|\,dt \right| \\
&\le K \left| \int_c^{c+h} 1 \cdot dt \right| \\
&= K|h| < \varepsilon, \text{ when } |h| < \frac{\varepsilon}{K} = \delta \text{ (say)}
\end{aligned}$$

$\therefore$ $\phi(t)$ is cont. at c.

Since c is arbitrary, ϕ is cont. on $[a,b]$.

Cor :— $\phi(t) = \int_t^b f(x)\,dx, \; a \le t < b$ with $\phi(b) = 0$ is cont. on $[a,b]$

H/W

Thm :— If f is cont. on $[a,b]$ and

$\phi(t) = \int^t f(x)\,dx, \quad a < t < b$ ...

$\phi(x) = \int_a^x f(t)\,dt, \quad a \le x \le b$, then
$$\phi'(t) = f(t), \quad \forall t \in [a,b].$$

$\Rightarrow$ Claim: $\displaystyle\lim_{h \to 0} \frac{\phi(c+h) - \phi(c)}{h} = f(c).$

Let $c \in [a,b]$ and $h > 0$ be s.t. $c + h \in [a,b]$.

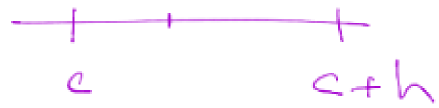
$$\phi(c+h) - \phi(c) = \int_c^{c+h} f(t)\,dt$$

$$\int_a^b f(x)\,dx = (b-a) f(\xi)$$
$$= (b-a)\mu \qquad \mu = f(\xi)$$

$$= h\, f(c + \theta h), \quad 0 \le \theta \le 1.$$

$$\Rightarrow \frac{\phi(c+h) - \phi(c)}{h} = f(c+\theta h)$$

$c + \theta h, \quad 0 \le \theta \le 1$



$\xi \in [a,b] \qquad \xi = a + (b-a)\theta, \quad 0 \le \theta \le 1$

$$\lim_{h \to 0} \frac{\phi(c+h) - \phi(c)}{h} = \lim_{h \to 0} f(c+\theta h)$$

$$\Rightarrow \phi'(c) = f(c).$$

c is arbitrary, $\phi'(t) = f(t), \ \forall t \in [a,b].$

* The continuity of f is only suff.

condition, it is not necessary cond.

Ex :—— $f : [0,1] \to \mathbb{R}$ s.t.

$$f(x) = \begin{cases} 3x^2 \sin \frac{1}{x^2} - 2\cos \frac{1}{x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

$$\phi(x) = \begin{cases} x^3 \sin \frac{1}{x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$\phi'(0) = \lim_{h \to 0} \frac{\phi(h) - \phi(0)}{h}$$

$$= \lim_{h \to 0} \frac{h^3 \sin \frac{1}{h^2}}{h} = 0 = f(0)$$

$$\phi'(x) = f(x), \ \forall x > 0.$$

$$\therefore \phi'(x) = f(x), \forall x \in [0,1].$$

f is not cont. at 0.