Fourier Analysis

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Fourier's theorem

- Any single-valued periodic function f(x), defined in the closed interval [-π,π] may be represented over the interval by the following trigonometric series

Fourier series

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)$$

where the expansion coefficients a_n and b_n are determined by ***Euler's formula*** :

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx \qquad b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx$$

The sufficient conditions for the convergence of the series (Dirichlet's theorem)

If for the interval [-π,π], the function f(x) is

- Bounded
- Single-valued
- Have only a finite number of extrema (maxima and minima)
- Can have finite number of finite discontinuities (piecewise continuous)

and if f(x+2π) = f(x), for all values of x outside [-π,π],

then

$$S_p(x) = \frac{1}{2}a_0 + \sum_{n=1}^{p}\left(a_n \cos nx + b_n \sin nx\right)$$

converges to f(x) as p tends to infinity at values of x for which f(x) is continuous.

Determination of Fourier coefficients

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx = \begin{cases} 0 & \text{for } n \neq m \\ 2\pi & \text{for } n = m = 0 \\ \pi & \text{for } n = m > 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx = \begin{cases} 0 & \text{for } n \neq m \\ 0 & \text{for } n = m = 0 \\ \pi & \text{for } n = m > 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(nx)\cos(mx)\,dx = 0 \quad \text{for all } n \text{ and } m$$

$$\int_{-\pi}^{\pi} \sin(nx)\,dx = 0 \quad \text{for all } n$$

$$\int_{-\pi}^{\pi} \cos(nx)\,dx = \begin{cases} 0 & \text{for } n > 0 \\ 2\pi & \text{for } n = 0 \end{cases}$$

- Multiplying $f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$

  by $\cos(mx)\,dx$ and carrying out the integration from

  x = -π to x = π, we will get a_0 and a_n

  and by $\sin(mx)\,dx$ and carrying out the integration from

- x = -π to x = π, we will get b_n

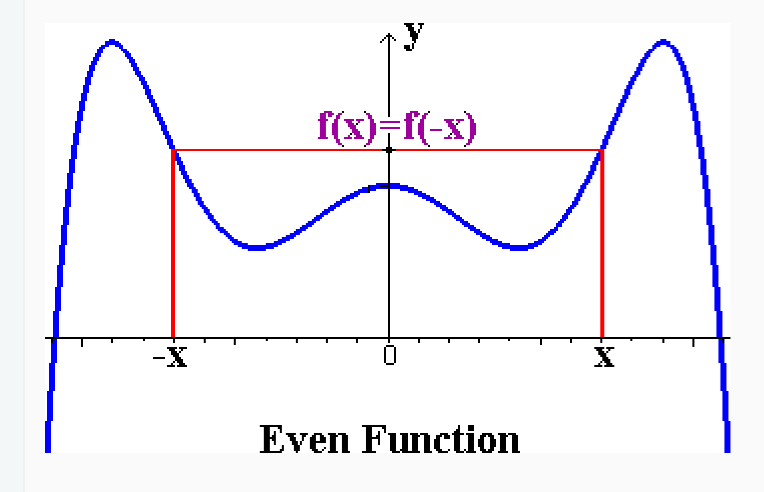
$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Fourier analysis of even function

- The function f(x) is even function f(x) = f(-x) in the range [-π,π]



$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$$

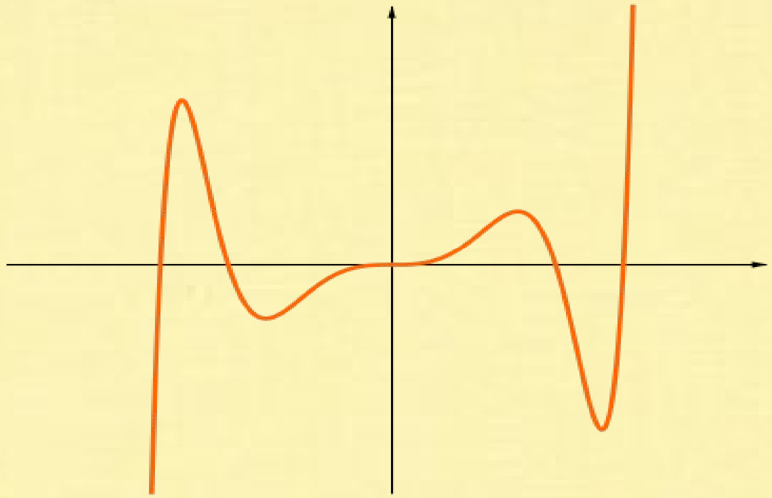
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0 \qquad \text{for all } n$$

$$\Rightarrow \quad f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

Fourier analysis of odd function

- The function f(x) is odd function f(x) = -f(-x) in the range [-π,π]



$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0, \qquad \text{for all } n$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

Change in interval of expansion

- So far the expansion has been restricted in the range [-π,π] → Change the interval to [-l,l]
- Let us take $f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos\phi x+b_n\sin\phi x)$

For $f(x)=f(x+2l), \phi=n\pi/l$

$$a_0=\frac{1}{l}\int_{-l}^{l} f(x)\,dx$$

$$a_n=\frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx \qquad \text{for } n=0,1,2,\ldots.$$

$$b_n=\frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx \qquad \text{for } n=1,2,3,\ldots.$$

→ $$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left[a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right]$$

Complex representation of Fourier series

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\left[\frac{e^{inx}+e^{-inx}}{2}\right]+b_n\left[\frac{e^{inx}-e^{-inx}}{2i}\right]=\sum_{n=-\infty}^{\infty}C_n e^{inx}$$

Fourier series in complex form

where $C_0=\frac{1}{2}a_0$

$$C_n=\frac{1}{2}(a_n-ib_n)$$

$$C_{-n}=\frac{1}{2}(a_n+ib_n)$$

Complex representation of Fourier series

$$\int_{-\pi}^{\pi} f(x)\, e^{-imx}\, dx = \sum_{-\infty}^{\infty} C_n \int_{-\pi}^{\pi} e^{-i(n-m)x}\, dx$$

$$= \sum_{-\infty}^{\infty} C_n \times 2\pi\, \delta_{mn} = 2\pi\, C_n$$

$$\text{since},\quad \delta_{mn} = \begin{cases} 1, & \text{when } n = m \\ 0, & \text{when } n \neq m \end{cases}$$

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$$

$$C_{-n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{inx}\, dx$$

$$C_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{2} a_0$$

Fourier analysis of f(x) = x, for $-\pi \leq x \leq \pi$

- The function f(x) is odd function f(x) = -f(-x) in the range [-π,π]

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx = 0, \quad \text{for all } n > 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx = \frac{2}{n}\cos(nx) = \frac{2}{n}(-1)^{n+1}$$

$$f(x) = \frac{\sum_{n=1}^{\infty}(-1)^{n+1}\sin(nx)}{n}$$

Obtain the Fourier series of the function

$$f(x)=\begin{cases} h, & \text{when } 0\le x<\pi \\ 0, & \text{when } -\pi\le x<0 \end{cases}$$

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)$$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{0}f(x)\,dx+\frac{1}{\pi}\int_{0}^{\pi}f(x)\,dx=\frac{1}{\pi}\int_{0}^{\pi}h\,dx=h$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{0}f(x)\cos nx\,dx+\frac{1}{\pi}\int_{-\pi}^{0}f(x)\cos nx\,dx=\frac{1}{\pi}\int_{0}^{\pi}h\cos nx\,dx=0$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{0}f(x)\sin nx\,dx+\frac{1}{\pi}\int_{-\pi}^{0}f(x)\sin nx\,dx=\frac{1}{\pi}\int_{0}^{\pi}h\sin nx\,dx$$

$$b_n=\frac{h}{\pi}\left[\frac{-\cos(nx)}{n}\right]=\begin{cases} 0, & \text{for } n \text{ even} \\ \dfrac{2h}{n\pi}, & \text{for } n \text{ odd} \end{cases}$$

$$f(x)=\frac{h}{2}+\frac{2h}{\pi}\sum_{n=1}^{\infty}\frac{\sin(nx)}{n} \qquad n\rightarrow odd$$

Obtain the Fourier series of the function

$$f(x)=\begin{cases} h, & \text{when } 0\le x<\pi \\ -h, & \text{when } \pi\le x\le 2\pi \end{cases}$$

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)$$

$$a_0=\frac{1}{\pi}\int_0^{\pi} h\,dx+\frac{1}{\pi}\int_{\pi}^{2\pi} -h\,dx=\frac{h}{\pi}[\pi-\pi]=0$$

$$a_n=\frac{1}{\pi}\int_0^{\pi} h\cos nx\,dx-\frac{1}{\pi}\int_{\pi}^{2\pi} h\cos nx\,dx=0$$

$$b_n=\frac{1}{\pi}\int_0^{\pi} h\sin nx\,dx-\frac{1}{\pi}\int_{\pi}^{2\pi} h\sin nx\,dx=\frac{2h}{n\pi}(1-\cos nx)$$

$$b_n=\begin{cases} 0, & \text{for } n \text{ even} \\ \dfrac{4h}{n\pi}, & \text{for } n \text{ odd} \end{cases}$$

$$f(x)=\frac{4h}{\pi}\sum_{n=1}^{\infty}\frac{\sin(nx)}{n} \qquad n\rightarrow odd$$

$$=\frac{4h}{\pi}\left[\sin x+\frac{\sin(3x)}{3}+\frac{\sin(5x)}{5}+....\right]$$

Obtain the Fourier series of the function

$$f(x)=\begin{cases}\sin x, & \text{when } 0\le x<\pi\\ 0, & \text{when } \pi\le x\le 2\pi\end{cases}$$

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)$$

$$a_0=\frac{1}{\pi}\int_0^{\pi} f(x)\,dx+\frac{1}{\pi}\int_{\pi}^{2\pi} f(x)\,dx=\frac{1}{\pi}\int_0^{\pi}\sin x\,dx=\frac{2}{\pi}$$

$$a_n=\frac{1}{\pi}\int_0^{\pi} f(x)\cos nx\,dx+\frac{1}{\pi}\int_{\pi}^{2\pi} f(x)\cos nx\,dx=\frac{1}{\pi}\int_0^{\pi}\sin x\cos nx\,dx$$

$$=\begin{cases}0, & \text{for } n \text{ odd}\\ \dfrac{-2}{\pi}\dfrac{1}{n^2-1}, & \text{for } n \text{ even}\end{cases}$$

Obtain the Fourier series of the function

$$f(x)=\begin{cases}\sin x, & \text{when } 0\le x<\pi\\ 0, & \text{when } \pi\le x\le 2\pi\end{cases}$$

$$a_n=\begin{cases}0, & \text{for } n \text{ odd}\\ \dfrac{-2}{\pi}\dfrac{1}{n^2-1}, & \text{for } n \text{ even}\end{cases}$$

$$b_n=\frac{1}{\pi}\int_0^{\pi} f(x)\sin nx\,dx+\frac{1}{\pi}\int_{\pi}^{2\pi} f(x)\sin nx\,dx=\frac{1}{\pi}\int_0^{\pi}\sin x\sin nx\,dx$$

$$=\begin{cases}0, & \text{for all } n \text{ except } 1\\ \dfrac{1}{2}, & \text{for } n=1\end{cases}$$

$$f(x)=\frac{1}{\pi}+\frac{1}{2}\sin x+\frac{2}{\pi}\sum_{n-even}\frac{\cos(nx)}{n^2-1}$$

Obtain the Fourier series of the function

$$f(x)=\begin{cases} \sin x, & \text{when } 0\le x<\pi \\ -\sin x, & \text{when } -\pi\le x\le 0 \end{cases}$$

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)$$

$$a_0=\frac{-1}{\pi}\int_{-\pi}^{0}\sin x\,dx+\frac{1}{\pi}\int_{0}^{\pi}\sin x\,dx=\frac{2}{\pi}\int_{0}^{\pi}\sin x\,dx=\frac{4}{\pi}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{0}-\sin x\cos nx\,dx+\frac{1}{\pi}\int_{0}^{\pi}\sin x\cos nx\,dx=\frac{2}{\pi}\int_{0}^{\pi}\sin x\cos nx\,dx$$

$$=\begin{cases} 0, & \text{for } n \text{ odd} \\ \dfrac{-4}{\pi}\dfrac{1}{n^2-1}, & \text{for } n \text{ even} \end{cases}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{0}-\sin x\sin nx\,dx+\frac{1}{\pi}\int_{0}^{\pi}\sin x\sin nx\,dx=\frac{2}{\pi}\int_{0}^{\pi}\sin x\sin nx\,dx=0$$

$$f(x)=\frac{2}{\pi}-\frac{4}{\pi}\sum_{n-even}\frac{\cos(nx)}{n^2-1}$$