Contents

1 Maxwell's equations

Electrodynamics is the branch of physics in which we discuss interactions among charges while they are in rest or in motion as well as interaction of electromagnetic (EM) wave with charges and related phenomena. The theory of electrodynamics is governed by three fundamental laws,

1. Gauss's law
2. Faraday's law
3. Ampere's circuital law

The mathematical form of these laws can be expressed both in differential and integral form. In free space (where there is no medium) the mathematical form of these laws;

Differential form :

$$\begin{aligned}
\nabla \cdot \boldsymbol{E} &= \frac{\rho}{\epsilon_0} \qquad \text{(Gauss's law)} \\
\nabla \times \boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t} \qquad \text{(Faraday's law)} \\
\nabla \times \boldsymbol{B} &= \mu_0 \boldsymbol{J} \qquad \text{(Ampere's circuital law)}
\end{aligned} \tag{1}$$

Apart from these three laws, there is a very special properties of magnetic field : *it's divergence is zero,*

$$\nabla \cdot \boldsymbol{B} = 0$$

This is the manifestation of the fact that magnetic monopole does not exist. Physically, divergence measures the outward flux of a vector field. Since there is no magnetic monopole, magnetic field vector originating from north pole must terminate to the south pole resulting net outward flux to be zero.

Integral form[1] :

$$\begin{aligned}
\oint_S \boldsymbol{E} \cdot d\boldsymbol{a} &= \frac{Q_{enc}}{\epsilon_0} \qquad \text{(Gauss's law)} \\
\oint_C \boldsymbol{E} \cdot d\boldsymbol{l} &= -\frac{d\Phi_B}{dt} \qquad \text{(Faraday's law)} \\
\oint_C \boldsymbol{B} \cdot d\boldsymbol{l} &= \mu_0 I_{enc} \qquad \text{(Ampere's circuital law)} \\
\oint_S \boldsymbol{B} \cdot d\boldsymbol{a} &= 0
\end{aligned} \tag{2}$$

Everything is fine except the Ampere's circuital law. The problem and its corrected form due to Maxwell is discussed in the following section.

[1]The integral forms can be obtained using divergence theorem and Stoke's theorem to the differential forms.

1.1 Maxwell's equations in free space

As already mentioned that something is wrong to the Ampere's circuital law. Consider the differential form of the Ampere's law and take divergence on both sides,

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{B}) = \mu_0 \boldsymbol{\nabla} \cdot \boldsymbol{J} \tag{3}$$

Now, **div.** of **curl** is always zero from vector identity; however the right hand side expression : $\boldsymbol{\nabla} \cdot \boldsymbol{J}$ is not always zero. It vanishes only for steady current where there is no accumulation of charges. In other words, inflow rate of charges is same as of the outflow rate such that total divergence of $\boldsymbol{J}$ turns out to be zero.

The general form of this law should be corrected in such a way that it takes into account the non-steady case also. Maxwell did this job in a very ingenious way. The Maxwell's corrected form has a great impact in the field of physical sciences bearing widespread consequences.

The continuity equation associated with the charge conservation,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{J} = 0 \tag{4}$$

From which using the Gauss's law we can find,

$$\boldsymbol{\nabla} \cdot \boldsymbol{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}(\epsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{E}) \tag{5}$$

That means,

$$\boldsymbol{\nabla} \cdot \left(\boldsymbol{J} + \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right) = 0 \tag{6}$$

So, if we transform,

$$\boldsymbol{J} \rightarrow \left(\boldsymbol{J} + \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right)$$

in the Ampere's circuital law then the problem will be resolved. The corrected form of the Ampere's circuital law proposed by Maxwell,

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \tag{7}$$

In this equation, divergence of both sides vanish.

The four basic laws of electrodynamics altogether are known as Maxwell's field equations. Before Maxwell's work, there was no clear picture about the connectivity of E and B fields. After Maxwell's work, the mathematical forms of the laws take symmetric form in between E and B field which is explicitly observed in Faraday's law and Ampere's law. In Faraday's law : changing magnetic field produces electric field and in Ampere's law (corrected form) : changing electric field produces magnetic field. Basically both $\boldsymbol{E}$ and $\boldsymbol{B}$ are same entities observed from different frame of references (which is clearly discussed in the context of relativity theory).

Finally we have the **Maxwell's equations in the free space** (no material medium)

Differential form :

$$\boxed{\begin{aligned}
&\text{(i)} && \nabla \cdot \boldsymbol{E}(\boldsymbol{r},t) = \frac{\rho(\boldsymbol{r},t)}{\epsilon_0} \\
&\text{(ii)} && \nabla \cdot \boldsymbol{B}(\boldsymbol{r},t) = 0 \\
&\text{(iii)} && \nabla \times \boldsymbol{E}(\boldsymbol{r},t) = -\frac{\partial \boldsymbol{B}(\boldsymbol{r},t)}{\partial t} \\
&\text{(iv)} && \nabla \times \boldsymbol{B}(\boldsymbol{r},t) = \mu_0 \boldsymbol{J}(\boldsymbol{r},t) + \mu_0\epsilon_0 \frac{\partial \boldsymbol{E}(\boldsymbol{r},t)}{\partial t}
\end{aligned}} \tag{8}$$

Where the differentiations are performed at $(\boldsymbol{r},t)$ point and the source terms i.e; charge density and current density are to be taken at the same point $(\boldsymbol{r},t)$. That means at those points where there is no charge density, $\nabla \cdot \boldsymbol{E}$ becomes zero.

Integral form :

$$\boxed{\begin{aligned}
&\text{(i)} && \oint_S \boldsymbol{E} \cdot d\boldsymbol{a} = \frac{Q_{enc}}{\epsilon_0} \\
&\text{(ii)} && \oint_S \boldsymbol{B} \cdot d\boldsymbol{a} = 0 \\
&\text{(iii)} && \oint_C \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{d\Phi_B}{dt} \\
&\text{(iv)} && \oint_C \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0 I_{enc} + \mu_0\epsilon_0 \frac{d\Phi_E}{dt}
\end{aligned}} \tag{9}$$

The term,

$$\epsilon_0 \frac{d\Phi_E}{dt}$$

is known as **displacement current** and the corresponding **displacement current density** is,

$$\boxed{\boldsymbol{J}_D = \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}} \tag{10}$$

The concept of displacement current can be understood with the example of parallel plate capacitor connected to a source. The source will charge the parallel plates of the capacitor and that

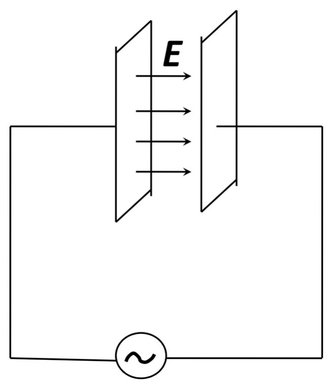

Figure 1:

eventually produces electric field across the plates. Suppose, $\sigma(t)$ be the surface charge density at the capacitor plates, then,

$$E = \frac{\sigma}{\epsilon_0}$$

The associated displacement current becomes,

$$\begin{aligned} I_D &= \epsilon_0 \frac{d\Phi_E}{dt} \\ &= \epsilon_0 \frac{d}{dt}(EA) \qquad ; \quad A = \text{area of the plates} \\ &= \frac{d}{dt}(\sigma A) = \frac{dQ}{dt} \end{aligned}$$

Now, $\frac{dQ}{dt}$ is the actual current flowing through the circuit. So, the displacement current in between two plates is identical to the physical current flowing through the circuit. Through the connecting wires, there is no displacement current and in reverse there is no physical current in between two plates of the capacitor.

Exercise 1. From the Coulomb's law, electrostatic field for a point charge (located at $r = 0$) is given by,

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{q}{4\pi\epsilon_0}\frac{\hat{\boldsymbol{r}}}{r^2}$$

Check the consistency of

$$\begin{aligned} \boldsymbol{\nabla}\cdot\boldsymbol{E} &= \frac{\rho}{\epsilon_0} \\ \text{and} \quad \oint_S \boldsymbol{E}\cdot d\boldsymbol{a} &= \frac{q}{\epsilon_0} \end{aligned} \tag{11}$$

Solution :

The given field $\boldsymbol{E}(\boldsymbol{r})$ is spherically symmetric due to its structure. Expression of divergence in spherical polar coordinate is given by,

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta E_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(E_\phi\right) \tag{12}$$

For the given field,

$$E_r = \frac{q}{4\pi\epsilon_0 r^2} \quad \text{and} \quad E_\theta = E_\phi = 0$$

Therefore,

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{q}{4\pi\epsilon_0 r^2}\right) = 0 \tag{13}$$

As expected, the divergence of $\boldsymbol{E}$ is to be zero for $r \neq 0$ because the charge is located exactly at $r = 0$. But it should be equal to ρ/ϵ_0 at $r = 0$ according to Maxwell's equation. This paradoxical situation arises due to the singularity arises from the point charge located at $\boldsymbol{r} = 0$; it is not possible to describe a point charge by means of any analytic expression of charge density $\rho(\boldsymbol{r})$ in form of,

$$q = \int_V \rho(\boldsymbol{r})d^3r$$

Special care should be taken in the divergence of $\boldsymbol{E}$ for point charge. Normal procedure to carry out the divergence fails at the point where the point charge is located.

Introduce the Dirac delta function in three dimension,

$$\begin{aligned} \delta(\boldsymbol{r}) &= 0 \qquad ; \text{for} \quad \boldsymbol{r} \neq 0 \\ &\neq 0 \qquad ; \text{for} \quad \boldsymbol{r} = 0 \end{aligned} \tag{14}$$

Along with its basic property,

$$\int_V f(\boldsymbol{r})\delta(\boldsymbol{r})d^3r = f(0) \tag{15}$$

With the help of delta function, the charge density associated with point charge located at $\boldsymbol{r} = 0$ can be expressed as,

$$\rho(\boldsymbol{r}) = q\delta(\boldsymbol{r}) \tag{16}$$

Then, divergence of $\boldsymbol{E}$ for point charge q at origin is defined as,

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{q}{4\pi\epsilon_0}\boldsymbol{\nabla} \cdot \left(\frac{\hat{\boldsymbol{r}}}{r^2}\right) = \frac{1}{\epsilon_0}q\delta(\boldsymbol{r}) \tag{17}$$

In the same spirit the divergence of inverse square function is defined as,

$$\boldsymbol{\nabla} \cdot \left(\frac{\hat{\boldsymbol{r}}}{r^2}\right) = 4\pi\delta(\boldsymbol{r}) \tag{18}$$

Consider the integral,

$$\oint_S \boldsymbol{E} \cdot d\boldsymbol{a} = \oint_S \frac{q}{4\pi\epsilon_0}\frac{\hat{\boldsymbol{r}}}{r^2} \cdot \underbrace{\left(\hat{\boldsymbol{r}}r^2 \sin\theta d\theta d\phi\right)}_{d\boldsymbol{a}} \tag{19}$$

$$= \frac{q}{4\pi\epsilon_0}\int_0^{\pi} \sin\theta \; d\theta \int_0^{2\pi} d\phi \tag{20}$$

$$\oint_S \boldsymbol{E} \cdot d\boldsymbol{a} = \frac{q}{\epsilon_0} \tag{21}$$

1.2 Maxwell's equations in matter

Everything is fine with the Maxwell's equation as prescribed from (i) to (iv) in equations (8) and (9) in free space, however inside matter, electric and magnetic fields produce electric and magnetic polarization to the material and as a result, bound charges and bound currents are formed on which we do not have explicit control. Furthermore, bound charges and bound currents give rise to new electric and magnetic field which superpose with the former fields (responsible for bound charges and currents). In this context it is convenient to rewrite the Maxwell's equations in terms of free charges and free currents on which we have direct control.

If the electric polarization vector is $\boldsymbol{P}$ then, bound charges associated with $\boldsymbol{P}$,

$$\rho_b = -\boldsymbol{\nabla} \cdot \boldsymbol{P} \tag{22}$$

and similarly bound current density $\boldsymbol{J}_b$ associated with the magnetic polarization vector $\boldsymbol{M}$ is,

$$\boldsymbol{J}_b = \boldsymbol{\nabla} \times \boldsymbol{M} \tag{23}$$

Now we add a new feature related to the time dependent case. Suppose, the polarization $\boldsymbol{P}$ is time dependent, then change of polarization gives rise to *polarization current density* $\boldsymbol{J}_p$. Physically polarization is the phenomena in which there is a separation of positive and negative charge density by very small spatial gap. Change of polarization in time means some amount of charge is flowing from positive to negative region or vice-versa that will results in flow of charge in time means current flow. Polarization gives rise to accumulation of bound surface charge density $\pm\sigma_b$ given by,

$$\sigma_b = \boldsymbol{P} \cdot \hat{\boldsymbol{n}}$$

where, $\hat{\boldsymbol{n}}$ is the surface normal unit vector. In case, if polarization changes with time, associated surface charge density also changes. The change of surface charge density can be interpreted as flow of charge to the surface resulting a current called *polarization current*. Total charge on a surface due to bound charge density σ_b is,

$$q_b = \int_S \sigma_b da$$

Then, polarization current becomes,

$$I_b = \frac{dq_b}{dt} = \frac{d}{dt}\int_S \sigma_b da = \int_S \frac{\partial \sigma_b}{\partial t} da = \int_S \frac{\partial \boldsymbol{P}}{\partial t} \cdot d\boldsymbol{a}$$

Therefore, the polarization current density identified as,

$$\boldsymbol{J}_p = \frac{\partial \boldsymbol{P}}{\partial t} \tag{24}$$

One thing is to be made clear that increase of bound surface charge density means flow of bound volume charge density to the surface and vise-verse such that total bound charge density (including volume charge and surface charge) remains constant. So, bound charge density should obey continuity equation. Let's check,

$$\frac{\partial \rho_b}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{J}_p = \frac{\partial}{\partial t}\underbrace{(-\boldsymbol{\nabla} \cdot \boldsymbol{P})}_{\rho_b} + \boldsymbol{\nabla} \cdot \underbrace{\frac{\partial \boldsymbol{P}}{\partial t}}_{\boldsymbol{J}_p} = 0$$

There is non violation of continuity equation.

Now **total volume charge density**,

$$\rho = \rho_f + \rho_b = \rho_f - \boldsymbol{\nabla} \cdot \boldsymbol{P}$$

and **total current density**,

$$\boldsymbol{J} = \boldsymbol{J}_f + \boldsymbol{J}_b + \boldsymbol{J}_p = \boldsymbol{J}_f + \boldsymbol{\nabla} \times \boldsymbol{M} + \frac{\partial \boldsymbol{P}}{\partial t}$$

where, f stands for *free* and b stands for *bound*

Hence, the Gauss's law becomes,

$$\begin{aligned} \boldsymbol{\nabla}\cdot\boldsymbol{E} &= \frac{\rho}{\epsilon_0} = \frac{1}{\epsilon_0}(\rho_f+\rho_b) = \frac{1}{\epsilon_0}(\rho_f - \boldsymbol{\nabla}\cdot\boldsymbol{P}) \\ \Rightarrow \qquad \boldsymbol{\nabla}\cdot(\epsilon_0\boldsymbol{E}+\boldsymbol{P}) &= \rho_f \\ \text{Or,} \qquad \boldsymbol{\nabla}\cdot\boldsymbol{D} &= \rho_f \end{aligned}$$

where, the *displacement vector* $\boldsymbol{D}$ is defined as,

$$\boldsymbol{D} = \epsilon_0\boldsymbol{E}+\boldsymbol{P}$$

The Ampere's law can be expressed as,

$$\begin{aligned} \boldsymbol{\nabla}\times\boldsymbol{B} &= \mu_0\boldsymbol{J} + \mu_0\epsilon_0\frac{\partial\boldsymbol{E}}{\partial t} \\ &= \mu_0(\boldsymbol{J}_f+\boldsymbol{J}_b+\boldsymbol{J}_p) + \mu_0\epsilon_0\frac{\partial\boldsymbol{E}}{\partial t} \\ &= \mu_0\left(\boldsymbol{J}_f + \boldsymbol{\nabla}\times\boldsymbol{M} + \frac{\partial\boldsymbol{P}}{\partial t}\right) + \mu_0\epsilon_0\frac{\partial\boldsymbol{E}}{\partial t} \\ \boldsymbol{\nabla}\times\left(\frac{1}{\mu_0}\boldsymbol{B}-\boldsymbol{M}\right) &= \boldsymbol{J}_f + \frac{\partial}{\partial t}(\epsilon_0\boldsymbol{E}+\boldsymbol{P}) \\ \boldsymbol{\nabla}\times\boldsymbol{H} &= \boldsymbol{J}_f + \frac{\partial\boldsymbol{D}}{\partial t} \end{aligned}$$

where,

$$\boldsymbol{H} = \frac{1}{\mu_0}\boldsymbol{B} - \boldsymbol{M}$$

Therefore, **Maxwell's equations inside material** become,

$$\boxed{\begin{aligned} &\text{(i)} \quad \boldsymbol{\nabla}\cdot\boldsymbol{D} = \rho_f \\ &\text{(ii)} \quad \boldsymbol{\nabla}\cdot\boldsymbol{B} = 0 \\ &\text{(iii)} \quad \boldsymbol{\nabla}\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t} \\ &\text{(iv)} \quad \boldsymbol{\nabla}\times\boldsymbol{H} = \boldsymbol{J}_f + \frac{\partial\boldsymbol{D}}{\partial t} \end{aligned}} \tag{25}$$

where,

$$\begin{aligned} \boldsymbol{D} &= \epsilon_0\boldsymbol{E}+\boldsymbol{P} \\ \boldsymbol{H} &= \frac{1}{\mu_0}\boldsymbol{B}-\boldsymbol{M} \end{aligned}$$

The equations takes simpler form when one deals with *linear medium.* In linear medium, polarization $\boldsymbol{P}$ is linearly proportional to the electric field $\boldsymbol{E}$ and magnetization $\boldsymbol{M}$ is linearly proportional to magnetic intensity $\boldsymbol{H}$[2]. Basically, when electric field strength and magnetic field strength is quite small, polarization and magnetization become proportional to field vectors. So, mathematically,

$$\boldsymbol{P} \propto \boldsymbol{E} \qquad \text{and} \qquad \boldsymbol{M} \propto \boldsymbol{H}$$

[2]Not to consider that $\boldsymbol{M} \propto \boldsymbol{B}$, because in any circumstances $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$ but in general $\boldsymbol{\nabla}\cdot\boldsymbol{M} \neq 0$.

The proportionality constant are taken in a clever form to express the Maxwell's equations in a simple form.

$$\boldsymbol{P} = \epsilon_0 \chi_e \boldsymbol{E} \qquad \text{and} \qquad \boldsymbol{M} = \chi_m \boldsymbol{H}$$

where, χ_e is called *electrical susceptibility* and χ_m is *magnetic susceptibility.*

Now, **for linear media**,

$$\boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P} = \epsilon_0 (1 + \chi_e) \boldsymbol{E} = \epsilon \boldsymbol{E}$$
$$\text{where,} \qquad \epsilon = \epsilon_0 (1 + \chi_e)$$

$$\boldsymbol{H} = \frac{1}{\mu_0} \boldsymbol{B} - \boldsymbol{M} = \frac{1}{\mu_0} \boldsymbol{B} - \chi_m \boldsymbol{H}$$
$$\Rightarrow \qquad \boldsymbol{H} = \frac{1}{\mu_0 (1 + \chi_m)} \boldsymbol{B} = \frac{1}{\mu} \boldsymbol{B}$$
$$\text{where,} \qquad \mu = \mu_0 (1 + \chi_m)$$

So, **for linear media** in summary,

$$\boxed{\begin{aligned} \boldsymbol{D} &= \epsilon \boldsymbol{E} \qquad \text{with} \qquad \epsilon = \epsilon_0 (1 + \chi_e) \\ \boldsymbol{H} &= \frac{1}{\mu} \boldsymbol{B} \qquad \text{with} \qquad \mu = \mu_0 (1 + \chi_m) \end{aligned}} \tag{26}$$

Therefore, from relations (25) and (26) **in linear media Maxwell's equations** ,

$$\boxed{\begin{aligned} &\text{(i)} \quad \boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho_f}{\epsilon} \\ &\text{(ii)} \quad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \\ &\text{(iii)} \quad \boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \\ &\text{(iv)} \quad \boldsymbol{\nabla} \times \boldsymbol{B} = \mu \boldsymbol{J}_f + \mu \epsilon \frac{\partial \boldsymbol{E}}{\partial t} \end{aligned}} \tag{27}$$

Note : In linear media, Maxwell's equation have the identical form to that in the vacuum; only replacements are $\epsilon_0 \rightarrow \epsilon = \epsilon_0(1 + \chi_e)$ and $\mu_0 \rightarrow \mu = \mu_0(1 + \chi_m)$. Furthermore, we have to consider only the free charges i.e; *free volume charge density* ρ_f and *free current density* $\boldsymbol{J}_f$ which are controllable.

Exercise 2. Consider a parallel plate capacitor is filled with linear dielectric of permittivity ϵ and permeability μ_0. Electrical conductivity of the dielectric is σ. Two ends of the capacitor are connected to an alternating voltage source $V(t) = V_0 \sin(\omega t)$. Find the ratio of conduction current I_c to the displacement current I_d.

Solution :

Suppose the are of the plates is A and distance between two plates is d, then total resistance of the dielectric material between two plates is,

$$R = \frac{d}{\sigma A}$$

So, total conduction current passing through the capacitor,

$$I_c(t) = \frac{V(t)}{R} = \frac{\sigma A V(t)}{d} = \frac{\sigma A V_0}{d} \sin(\omega t)$$

Electric field between two plates,

$$E(t) = \frac{V(t)}{d}$$

Displacement current density,

$$J_d = \epsilon \frac{\partial E(t)}{\partial t} = \frac{\epsilon}{d} \frac{\partial V(t)}{\partial t} = \frac{\epsilon \omega V_0}{d} \cos(\omega t)$$

Displacement current,

$$I_d(t) = A J_d(t) = \frac{A \epsilon \omega V_0}{d} \cos(\omega t)$$

Therefore,

$$\frac{I_{d,max}}{I_{c,max}} = \frac{\omega \epsilon}{\sigma}$$

1.3 Boundary conditions

Going from one dielectric medium to another it is interesting to see how the fields are connected across the boundary or interface between two media. This study is very much important to understand the phenomena of reflection and transmission of electromagnetic field at some surface separating two media. To analyze the connectivity of fields at boundary of two **linear media** it is easier to adopt the integral form of Maxwell's relations (i)-(iv) of (25).

The **integral form of equations (25)**,

$$\boxed{\begin{array}{ll} \text{(i)} & \oint_S \boldsymbol{D} \cdot d\boldsymbol{a} = Q_{f,enc.} \\ \text{(ii)} & \oint_S \boldsymbol{B} \cdot d\boldsymbol{a} = 0 \\ \text{(iii)} & \oint_C \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{d}{dt} \int_S \boldsymbol{B} \cdot d\boldsymbol{a} \\ \text{(iv)} & \oint_C \boldsymbol{H} \cdot d\boldsymbol{l} = I_{f,enc.} + \frac{d}{dt} \int_S \boldsymbol{D} \cdot d\boldsymbol{a} \end{array}} \tag{28}$$

Refer to figure 2 (left), a surface is separating two media medium-1 and medium-2. Consider a Gaussian surface of cylindrical shape with infinitesimal small height extended both side of the surface. Electric flux through the curved surface is negligible due to its vanishingly small height; only the flux need to be considered for two basal surfaces of area A (say). Then from relation (28-i),

$$(\boldsymbol{D}_1 \cdot \hat{\boldsymbol{n}} - \boldsymbol{D}_2 \cdot \hat{\boldsymbol{n}})A = Q_{f,enc} = \sigma_f A$$

$$\text{or,} \quad D_1^{\perp} - D_2^{\perp} = \sigma_f$$

$$\text{or,} \quad \epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f \qquad \text{(for linear media)}$$

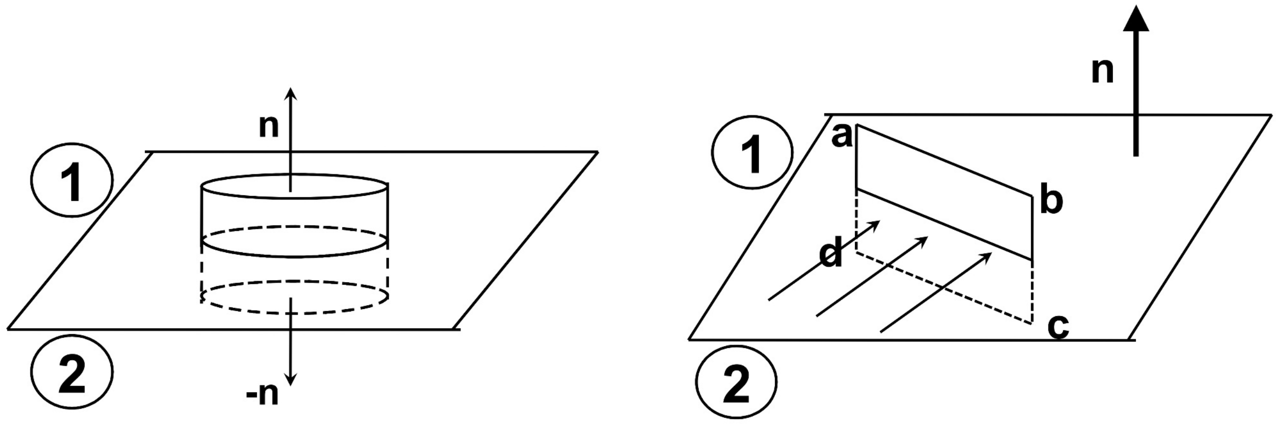

Figure 2:

In the same technique, using (28-ii),

$$B_1^{\perp} - B_2^{\perp} = 0$$

Above two relations gives us the connections of fields of their perpendicular components to the boundary surface.

The boundary conditions related to the parallel components of the fields can be developed with the help of figure-2 (right). Consider a Amperean loop *abcda* going across the surface with vanishingly small height extended across the surface in both sides. Now consider the Maxwell's equation (28-iii),

$$\oint_C \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{d}{dt} \int_S \boldsymbol{B} \cdot d\boldsymbol{a}$$

where the line integral goes around the Amperean loop $a \to b \to c \to d \to a$ and surface area is enclosed by this loop. According to consideration, width of the loop is vanishingly small i.e; line segment ad and bc tends to zero resulting magnetic flux to be zero through the loop area. Furthermore, due to same reason, line integration of $\boldsymbol{E}$ along $b \to c$ and $d \to a$ vanishes. So, the above equation becomes,

$$\int_a^b \boldsymbol{E_1} \cdot d\boldsymbol{l} + \int_c^d \boldsymbol{E_2} \cdot d\boldsymbol{l} = 0$$

$$\text{or,} \quad \int_a^b \boldsymbol{E_1} \cdot d\boldsymbol{l} - \int_d^c \boldsymbol{E_2} \cdot d\boldsymbol{l} = 0$$

$$\text{or,} \quad \int_a^b (\boldsymbol{E}_1 - \boldsymbol{E}_2) \cdot d\boldsymbol{l} = 0 \qquad \text{; since segment lengths } ab = dc$$

This is true for any arbitrary line element parallel to the surface, therefore, we have,

$$\boldsymbol{E}_1^{\parallel} - \boldsymbol{E}_2^{\parallel} = 0 \qquad \text{across the boundary.}$$

Now consider relation (28-iv),

$$\oint_C \boldsymbol{H} \cdot d\boldsymbol{l} = I_{f,enc.} + \frac{d}{dt} \int_S \boldsymbol{D} \cdot d\boldsymbol{a}$$

for the Amperean loop $a \to b \to c \to d \to a$ depicted in figure-2(right). According to above discussions, surface integration of $\boldsymbol{D}$ vanishes for this loop area. Also line integration of $\boldsymbol{H}$ becomes zero for line segment $b \to c$ and $d \to a$. So, the above relation becomes,

$$\int_a^b \boldsymbol{H_1} \cdot d\boldsymbol{l} + \int_c^d \boldsymbol{H_2} \cdot d\boldsymbol{l} = I_{f,enc.}$$

Suppose the surface caries a *free surface current density*[3] $\boldsymbol{K}_f$ which gives the enclosed free current $I_{f,enc.}$ passing through the loop. If $\hat{\boldsymbol{l}}$ is the unit vector along $a \to b$ then $d\boldsymbol{l} = \hat{\boldsymbol{l}}dl$. Hence, the unit vector which is perpendicular to $a \to b$ line segment and parallel to surface is,

$$\hat{\boldsymbol{t}} = \hat{\boldsymbol{n}} \times \hat{\boldsymbol{l}}$$

Here,

$$I_{f,enc.} = \int_a^b \boldsymbol{K}_f \cdot \hat{\boldsymbol{t}}\, dl = \int_a^b \boldsymbol{K}_f \cdot (\hat{\boldsymbol{n}} \times \hat{\boldsymbol{l}}) dl = \int_a^b (\boldsymbol{K}_f \times \hat{\boldsymbol{n}}) \cdot d\boldsymbol{l}$$

Therefore from the above equations we have,

$$\begin{aligned} &\int_a^b \boldsymbol{H}_1 \cdot d\boldsymbol{l} + \int_c^d \boldsymbol{H}_2 \cdot d\boldsymbol{l} = I_{f,enc.} = \int_a^b (\boldsymbol{K}_f \times \hat{\boldsymbol{n}}) \cdot d\boldsymbol{l} \\ \text{or,} \quad &\int_a^b \boldsymbol{H}_1 \cdot d\boldsymbol{l} - \int_d^c \boldsymbol{H}_2 \cdot d\boldsymbol{l} = \int_a^b (\boldsymbol{K}_f \times \hat{\boldsymbol{n}}) \cdot d\boldsymbol{l} \\ \text{or,} \quad &\int_a^b (\boldsymbol{H}_1 - \boldsymbol{H}_2) \cdot d\boldsymbol{l} = \int_a^b (\boldsymbol{K}_f \times \hat{\boldsymbol{n}}) \cdot d\boldsymbol{l} \end{aligned} \tag{29}$$

This is true for any arbitrary line element parallel to the surface, therefore, we have,

$$\boldsymbol{H}_1^{\parallel} - \boldsymbol{H}_2^{\parallel} = \boldsymbol{K}_f \times \hat{\boldsymbol{n}} \qquad \text{; across the boundary.}$$

Here we **summarize the boundary conditions of fields at the boundary separating two media**

$$\boxed{\begin{aligned} \text{(i)} \quad & D_1^{\perp} - D_2^{\perp} = \sigma_f \\ \text{(ii)} \quad & B_1^{\perp} - B_2^{\perp} = 0 \\ \text{(iii)} \quad & \boldsymbol{E}_1^{\parallel} - \boldsymbol{E}_2^{\parallel} = 0 \\ \text{(iv)} \quad & \boldsymbol{H}_1^{\parallel} - \boldsymbol{H}_2^{\parallel} = \boldsymbol{K}_f \times \hat{\boldsymbol{n}} \end{aligned}} \tag{30}$$

For linear media, the boundary conditions become,

$$\boxed{\begin{aligned} \text{(i)} \quad & \epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f \\ \text{(ii)} \quad & B_1^{\perp} - B_2^{\perp} = 0 \\ \text{(iii)} \quad & \boldsymbol{E}_1^{\parallel} - \boldsymbol{E}_2^{\parallel} = 0 \\ \text{(iv)} \quad & \frac{1}{\mu_1}\boldsymbol{B}_1^{\parallel} - \frac{1}{\mu_2}\boldsymbol{B}_2^{\parallel} = \boldsymbol{K}_f \times \hat{\boldsymbol{n}} \end{aligned}} \tag{31}$$

[3]Surface current density is the amount of current flowing over the surface and passing through per unit line segment which is perpendicular to flow and placed on the surface.

1.4 Electromagnetic wave

Maxwell's equations lead to wave equation for electric field $\boldsymbol{E}(\boldsymbol{r},t)$ and magnetic field $\boldsymbol{B}(\boldsymbol{r},t)$. This is the most important outcome of Maxwell's correction term in Ampere's circuital law.

In vacuum, Maxwell's equations are,

$$
\begin{aligned}
&\text{(i)} && \boldsymbol{\nabla}\cdot\boldsymbol{E}(\boldsymbol{r},t)=0\\
&\text{(ii)} && \boldsymbol{\nabla}\cdot\boldsymbol{B}(\boldsymbol{r},t)=0\\
&\text{(iii)} && \boldsymbol{\nabla}\times\boldsymbol{E}(\boldsymbol{r},t)=-\frac{\partial\boldsymbol{B}(\boldsymbol{r},t)}{\partial t}\\
&\text{(iv)} && \boldsymbol{\nabla}\times\boldsymbol{B}(\boldsymbol{r},t)=\mu_0\epsilon_0\frac{\partial\boldsymbol{E}(\boldsymbol{r},t)}{\partial t}
\end{aligned}
$$

Now taking curl of (iii),

$$
\begin{aligned}
&& \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{E}) &= -\frac{\partial}{\partial t}(\boldsymbol{\nabla}\times\boldsymbol{B})\\
\text{or,} && -\nabla^2\boldsymbol{E}+\boldsymbol{\nabla}\underbrace{(\boldsymbol{\nabla}\cdot\boldsymbol{E})}_{0} &= -\frac{\partial}{\partial t}\left(\mu_0\epsilon_0\frac{\partial\boldsymbol{E}}{\partial t}\right)\\
\text{or,} && \nabla^2\boldsymbol{E}-\mu_0\epsilon_0\frac{\partial^2\boldsymbol{E}}{\partial t^2} &= 0
\end{aligned}
$$

Similarly, taking curl of (iv) and using Maxwell's equations, one can establish similar kind of differential equation for $\boldsymbol{B}$.

Hence we have **electromagnetic wave equations in vacuum**,

$$
\boxed{\begin{aligned}
&\nabla^2\boldsymbol{E}-\mu_0\epsilon_0\frac{\partial^2\boldsymbol{E}}{\partial t^2}=0\\
&\text{and}\\
&\nabla^2\boldsymbol{B}-\mu_0\epsilon_0\frac{\partial^2\boldsymbol{B}}{\partial t^2}=0
\end{aligned}} \tag{32}
$$

► The standard form of wave equation associated with disturbance $\psi(\boldsymbol{r},t)$ is,

$$
\nabla^2\psi(\boldsymbol{r},t)-\frac{1}{v^2}\frac{\partial^2\psi(\boldsymbol{r},t)}{\partial t^2}=0 \tag{33}
$$

where, v is the speed of propagation of wave.

► The most important observation is that in electromagnetic wave, both electric field as well as magnetic field exhibit wave nature. The electromagnetic wave in vacuum travels with constant speed determined by two universal constants μ_0 and ϵ_0 as,

$$
\boxed{c=\frac{1}{\sqrt{\mu_0\epsilon_0}}=3\times10^8\ m/s}
$$

Which is identical to the experimentally determined speed of visible light in vacuum. So, light is nothing but electromagnetic wave specified by a range of frequency of electromagnetic wave. Furthermore, the speed is independent of particular choice of frame of reference. That means, Galilean velocity addition is no longer valid for electromagnetic wave. The speed remain constant at the value of $3 \times 10^8\ m/s$ (in vacuum) whether the source and observer are in relative motion.

► In **charge free and current free region of linear dielectric**, the Maxwell's equation take the same form that of in vacuum with replacement of $\epsilon_0 \to \epsilon = \epsilon_0(1+\chi_e)$ and $\mu_0 \to \mu = \mu_0(1+\chi_m)$. That means wave equations in such situation become,

$$\boxed{\begin{array}{l} \nabla^2 \boldsymbol{E} - \mu\epsilon \dfrac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0 \\ \text{and} \\ \nabla^2 \boldsymbol{B} - \mu\epsilon \dfrac{\partial^2 \boldsymbol{B}}{\partial t^2} = 0 \end{array}} \tag{34}$$

The wave propagation speed is identified as,

$$\boxed{v = \frac{1}{\sqrt{\mu\epsilon}}}$$

Hence, it is interesting to identify the **refractive index of a medium** as,

$$\boxed{n = \frac{c}{v} = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}}}$$

1.4.1 Solution of wave equation

The wave equation for $\boldsymbol{E}$ and $\boldsymbol{B}$ being identical, both will give identical solution. So, we seek solution for any one of them say for $\boldsymbol{E}(\boldsymbol{r}, t)$.

Consider Cartesian system $(x,\ y,\ z)$ in which,

$$\boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{E}(x, y, z, t) = \hat{\boldsymbol{x}} E_x(x, y, z, t) + \hat{\boldsymbol{y}} E_y(x, y, z, t) + \hat{\boldsymbol{z}} E_z(x, y, z, t)$$

The wave equation,

$$\nabla^2 \boldsymbol{E} = \frac{1}{c^2} \frac{\partial^2 \boldsymbol{E}}{\partial t^2}$$

$$\text{or,} \qquad \nabla^2 (\hat{\boldsymbol{x}} E_x + \hat{\boldsymbol{y}} E_y + \hat{\boldsymbol{z}} E_z) = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} (\hat{\boldsymbol{x}} E_x + \hat{\boldsymbol{y}} E_y + \hat{\boldsymbol{z}} E_z)$$

Therefore, for each Cartesian component, wave equation shares the same form,

$$\nabla^2 E_j(x, y, z, t) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} E_j(x, y, z, t) = 0 \qquad ; \quad \boxed{j = x,\ y,\ z}$$

or,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) E_j - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} E_j = 0$$

The equation can be solved by the method of separation of variable. Consider,

$$E_j(x, y, z, t) = f_1(x) f_2(y) f_3(z) f_4(t)$$

Then the above differential equation becomes,

$$f_2 f_3 f_4 \frac{d^2 f_1}{dx^2} + f_1 f_3 f_4 \frac{d^2 f_2}{dy^2} + f_1 f_2 f_4 \frac{d^2 f_3}{dz^2} - \frac{1}{c^2} f_1 f_2 f_3 \frac{d^2 f_4}{dt^2} = 0$$

$$\text{or,} \qquad \frac{1}{f_1(x)} \frac{d^2 f_1(x)}{dx^2} + \frac{1}{f_2(y)} \frac{d^2 f_2(y)}{dy^2} + \frac{1}{f_3(z)} \frac{d^2 f_3(z)}{dz^2} - \frac{1}{c^2 f_4(t)} \frac{d^2 f_4(t)}{dt^2} = 0$$

This is true for all value of x, y, z, t, therefore it is possible only when each terms are equal to some constants. Consider,

$$\frac{1}{f_1(x)} \frac{d^2 f_1(x)}{dx^2} = -k_x^2$$
$$\frac{1}{f_2(y)} \frac{d^2 f_2(y)}{dy^2} = -k_y^2$$
$$\frac{1}{f_3(z)} \frac{d^2 f_3(z)}{dz^2} = -k_z^2$$
$$\frac{1}{c^2} \frac{1}{f_4(t)} \frac{d^2 f_4(t)}{dt^2} = -k^2$$

where,

$$k^2 = k_x^2 + k_y^2 + k_z^2$$

The differential equations represents oscillatory behaviour. The solutions can be expressed as,

$$f_1(x) = a_1 \exp(\pm i x k_x) \ , \ f_2(y) = a_2 \exp(\pm i y k_y) \ , \ f_3(z) = a_3 \exp(\pm i z k_z) \ , \ f_4(t) = a_4 \exp(\pm i c k)$$

Therefore, the solution for $E_j(x, y, z, t)$,

$$\begin{aligned} E_j(x, y, z, t) &= f_1(x) f_2(y) f_3(z) f_4(t) \\ &= a_1 a_2 a_3 a_4 \exp(\pm i x k_x \pm i y k_y \pm i z k_z \pm i t c k) \\ &= E_{0j} \exp[i(\pm \boldsymbol{k} \cdot \boldsymbol{r} \pm \omega t)] \end{aligned}$$

where, we have introduced the **wave vector**

$$\boxed{\boldsymbol{k} = \hat{\boldsymbol{x}} k_x + \hat{\boldsymbol{y}} k_y + \hat{\boldsymbol{z}} k_z}$$

and also we guess the **angular frequency** ω as,

$$\boxed{\omega = c|\boldsymbol{k}| = ck}$$

That means for each Cartesian component the solution of wave equation gives rise to the plane wave solution

$$E_j(x, y, z, t) = E_{0j} \exp[i(\pm \boldsymbol{k} \cdot \boldsymbol{r} \pm \omega t)] \qquad ; \text{for } j = x,\ y,\ z$$

So, for $\boldsymbol{E}(\boldsymbol{r}, t)$,

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{r}, t) &= \hat{\boldsymbol{x}} E_x(x, y, z, t) + \hat{\boldsymbol{y}} E_y(x, y, z, t) + \hat{\boldsymbol{z}} E_z(x, y, z, t) \\ &= (\hat{\boldsymbol{x}} E_{0x} + \hat{\boldsymbol{y}} E_{0y} + \hat{\boldsymbol{z}} E_{0z}) \exp[i(\pm \boldsymbol{k} \cdot \boldsymbol{r} \pm \omega t)] \\ &= \boldsymbol{E}_0 \exp[i(\pm \boldsymbol{k} \cdot \boldsymbol{r} \pm \omega t)] \end{aligned}$$

where $\boldsymbol{E}_0 = (\hat{\boldsymbol{x}}E_{0x} + \hat{\boldsymbol{y}}E_{0y} + \hat{\boldsymbol{z}}E_{0z})$ is the amplitude of wave.
Solving the wave equation for $\boldsymbol{B}(\boldsymbol{r},t)$, one can obtain the similar expression.

► Conventionally, **plane progressive and monochromatic electromagnetic wave propagating along the direction of wave vector $\boldsymbol{k}$** is represented as,

$$\boxed{\begin{aligned}\boldsymbol{E}(\boldsymbol{r},t) &= \boldsymbol{E}_0 \exp[i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)] \\ \boldsymbol{B}(\boldsymbol{r},t) &= \boldsymbol{B}_0 \exp[i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]\end{aligned}} \tag{35}$$

► **Wave equation in curvilinear system :** This is to be noted that it is very much complicated to solve vector wave equation in curvilinear coordinate systems (e.g. spherical polar). The Laplacian ∇^2 of a vector can not be decomposed according the vector components along the axes. For example, in arbitrary curvilinear system (u_1, u_2, u_3), one can express

$$\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E}(u_1,u_2,u_3,t) = \hat{\boldsymbol{e}}_1 E_1(u_1,u_2,u_3,t) + \hat{\boldsymbol{e}}_2 E_2(u_1,u_2,u_3,t) + \hat{\boldsymbol{e}}_3 E_3(u_1,u_2,u_3,t)$$

In cuvilinear system in general for i-th component,

$$\left(\nabla^2 \boldsymbol{E}\right)_i \neq \nabla^2 E_i \qquad ; \text{ for } \quad i = 1,2,3$$

This is due to the fact that in cuvilinear system, direction of unit vectors varies from point to point, so, spatial derivative of the unit vectors are non-vanishing in general. That means the expression $\nabla^2(\hat{\boldsymbol{e}}_i E_i)$ can not be expressed as $\hat{\boldsymbol{e}}_i \nabla^2 E_i$. So, in curvilinear system

$$\nabla^2 \boldsymbol{E}(u_1,u_2,u_3,t) \neq \sum_{i=1}^{3} \hat{\boldsymbol{e}}_i \nabla^2 E_i(u_1,u_2,u_3,t)$$

However, such decomposition is possible in Cartesian system, which makes the vector wave equation solvable in Cartesian system.

1.4.2 Elecromagnetic fields and wave vector are mutually perpendicular

So, far we have obtained the plane wave solution for electromagnetic wave as represented by (35). Still we don't know the nature of the wave; whether it is transverse or longitudinal. Further, Maxwell's equations tell something about the connection of electric and magnetic fields in wave. Following we address these queries. To proceed through these topics, it may be helpful to go through the following Exercise - 3

Exercise 3. Suppose three Cartesian axes are reassigned as $(x,y,z) \to (x_1,x_2,x_3)$, then show that,

$$\boxed{\frac{\partial}{\partial x_j}\exp[i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)] = ik_j \exp[i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)] \qquad ; \text{for } j = 1,2,3} \tag{36}$$

Solution :

$$\boldsymbol{k} = \hat{\boldsymbol{x}}_1 k_1 + \hat{\boldsymbol{x}}_2 k_2 + \hat{\boldsymbol{x}}_3 k_3 = \sum_{l=1}^{3} \hat{\boldsymbol{x}}_l k_l$$

$$\text{and} \qquad \boldsymbol{r} = \hat{\boldsymbol{x}}_1 x_1 + \hat{\boldsymbol{x}}_2 x_2 + \hat{\boldsymbol{x}}_3 x_3 = \sum_{m=1}^{3} \hat{\boldsymbol{x}}_m x_m$$

Therefore,

$$\boldsymbol{k} \cdot \boldsymbol{r} = \sum_{l=1}^{3} \sum_{m=1}^{3} (\hat{\boldsymbol{x}}_l k_l) \cdot (\hat{\boldsymbol{x}}_m x_m) = \sum_{l=1}^{3} \sum_{m=1}^{3} k_l x_m \underbrace{\hat{\boldsymbol{x}}_l \cdot \hat{\boldsymbol{x}}_m}_{\delta_{lm}} = \sum_{l=1}^{3} \sum_{m=1}^{3} x_m k_l \delta_{lm} = \sum_{l=1}^{3} x_l k_l$$

Now,

$$\begin{aligned}
\frac{\partial}{\partial x_j} \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)] &= \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)] \ \frac{\partial}{\partial x_j} [i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)] \\
&= \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)] \ \frac{\partial}{\partial x_j} \left(i \sum_{l=1}^{3} x_l k_l \right) \qquad ; \omega \text{ independent of } x \\
&= \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)] \ \left(i \sum_{l=1}^{3} k_l \delta_{jl} \right) \qquad ; \frac{\partial x_l}{\partial x_j} = \delta_{jl} \\
\frac{\partial}{\partial x_j} \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)] &= i k_j \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)]
\end{aligned}$$

Hence proved

In Cartesian system (x_1, x_2, x_3), the $\boldsymbol{E}(\boldsymbol{r}, t)$ field for plane progressive electromagnetic wave can be expressed as,

$$\boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{E}_0 \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)] = \underbrace{\left(\sum_{j=1}^{3} \hat{\boldsymbol{x}}_j E_{0j} \right)}_{\boldsymbol{E}_0} \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)]$$

Now evaluate the divergence for the above field. In Cartesian system it is,

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \boldsymbol{E}(\boldsymbol{r}, t) &= \sum_{j=1}^{3} \hat{\boldsymbol{x}}_j \frac{\partial}{\partial x_j} \cdot \boldsymbol{E}_0 \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)] \\
&= \sum_{j=1}^{3} \hat{\boldsymbol{x}}_j \frac{\partial}{\partial x_j} \cdot \sum_{l=1}^{3} \hat{\boldsymbol{x}}_l E_{0l} \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)] \\
&= \sum_{j=1}^{3} E_{0j} \frac{\partial}{\partial x_j} \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)] \qquad ; \hat{\boldsymbol{x}}_j \cdot \hat{\boldsymbol{x}}_l = \delta_{jl} \\
&= \sum_{j=1}^{3} E_{0j} (i k_j) \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)] \qquad ; \text{ using Exercise - 3} \\
\boldsymbol{\nabla} \cdot \boldsymbol{E}(\boldsymbol{r}, t) &= i \boldsymbol{k} \cdot \boldsymbol{E}(\boldsymbol{r}, t)
\end{aligned}$$

Now, consider the Maxwell equation in charge free region,

$$\boldsymbol{\nabla} \cdot \boldsymbol{E}(\boldsymbol{r}, t) = 0$$

Therefore, we have obtained that

$$\boxed{\boldsymbol{\nabla} \cdot \boldsymbol{E}(\boldsymbol{r}, t) = i\boldsymbol{k} \cdot \boldsymbol{E}(\boldsymbol{r}, t) = 0} \tag{37}$$

for electromagnetic field.

Since, the vector dot product of wave vector $\boldsymbol{k}$ and electric field $\boldsymbol{E}$ is zero, that means **electric field vibrates along perpendicular direction to the propagation direction $\hat{\boldsymbol{k}}$**.

Similarly, starting from the expression $\boldsymbol{B}(\boldsymbol{r}, t) = \boldsymbol{B}_0 \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)]$ and using the Maxwell equation $\boldsymbol{\nabla} \cdot \boldsymbol{B}(\boldsymbol{r}, t) = 0$, one can establish,

$$\boxed{\boldsymbol{\nabla} \cdot \boldsymbol{B}(\boldsymbol{r}, t) = i\boldsymbol{k} \cdot \boldsymbol{B}(\boldsymbol{r}, t) = 0} \tag{38}$$

So, it is obvious that **magnetic field vibrates perpendicular to propagation direction $\hat{\boldsymbol{k}}$**.

► Both electric field $\boldsymbol{E}(\boldsymbol{r}, t)$ and magnetic field $\boldsymbol{B}(\boldsymbol{r}, t)$ is perpendicular to propagation direction $\hat{\boldsymbol{k}}$. So, the **electromagnetic wave is transverse in nature**.

To establish the relation between $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{B}(\boldsymbol{r}, t)$ in electromagnetic field, consider the Maxwell equation,

$$\boldsymbol{\nabla} \times \boldsymbol{E}(\boldsymbol{r}, t) = -\frac{\partial}{\partial t}\boldsymbol{B}(\boldsymbol{r}, t)$$

Before proceed to calculate curl of $\boldsymbol{E}(\boldsymbol{r}, t)$, let's be accustomed with **Levi-Civita symbol $\boldsymbol{\mathcal{E}_{lmn}}$** which helps to carry out vector cross product calculations in compact form. It is defined as,

$$\boxed{\mathcal{E}_{lmn} = \begin{cases} 1 & ; \text{ for } \{lmn\} = \{123\}\ ,\ \{231\}\ ,\ \{312\} \quad \text{(Clockwise combination)} \\ -1 & ; \text{ for } \{lmn\} = \{132\}\ ,\ \{321\}\ ,\ \{213\} \quad \text{(Anti-clockwise combination)} \\ 0 & ; \text{ for any repeated indices} \end{cases}} \tag{39}$$

The value of Levi-Civita symbol $\mathcal{E}_{lmn}$ can be keep in mind with the help of the diagram 3. For clockwise permutation of the indices it is +1 (shown in figure - 3(left)) and it is -1 for anti-clockwise permutation (shown in figure - 3(right)). Suppose two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are represented in Cartesian

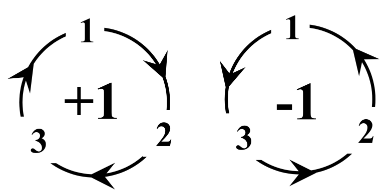

Figure 3:

system (x_1, x_2, x_3) as,

$$\boldsymbol{a} = \sum_{j=1}^{3} \hat{\boldsymbol{x}}_j a_j \qquad \text{and} \qquad \boldsymbol{b} = \sum_{j=1}^{3} \hat{\boldsymbol{x}}_j b_j$$

then using Levi-Civita symbol,

$$\boxed{\boldsymbol{a} \times \boldsymbol{b} = \sum_{l,m,n=1}^{3} \mathcal{E}_{lmn}\ \hat{\boldsymbol{x}}_l a_m b_n} \tag{40}$$

Now, for electromagnetic wave, the electric field in Cartesian system (x_1, x_2, x_3) represented by,

$$\begin{aligned}
\boldsymbol{E}(\boldsymbol{r}, t) = \sum_{j=1}^{3} \hat{\boldsymbol{x}}_j E_j(\boldsymbol{r}, t) &= \sum_{j=1}^{3} \hat{\boldsymbol{x}}_j \underbrace{E_{0j} \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)]}_{E_j(\boldsymbol{r},t)} \\
&= \underbrace{\left(\sum_{j=1}^{3} \hat{\boldsymbol{x}}_j E_{0j} \right)}_{\boldsymbol{E}_0} \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)] \\
&= \boldsymbol{E}_0 \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)]
\end{aligned}$$

The vector differential operator in Cartesian system (x_1, x_2, x_3) is given by,

$$\boldsymbol{\nabla} = \sum_{j=1}^{3} \hat{\boldsymbol{x}}_j \frac{\partial}{\partial x_j}$$

So, with the help of equation (40) in Cartesian system (x_1, x_2, x_3),

$$\begin{aligned}
\boldsymbol{\nabla} \times \boldsymbol{E}(\boldsymbol{r}, t) &= \sum_{l,m,n=1}^{3} \mathcal{E}_{lmn}\ \hat{\boldsymbol{x}}_l \frac{\partial}{\partial x_m} E_n(\boldsymbol{r}, t) \qquad ; \quad E_n(\boldsymbol{r}, t) = E_{0n} \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)] \\
&= \sum_{l,m,n=1}^{3} \mathcal{E}_{lmn}\ \hat{\boldsymbol{x}}_l E_{0n} \frac{\partial}{\partial x_m} \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)] \\
&= \sum_{l,m,n=1}^{3} \mathcal{E}_{lmn}\ \hat{\boldsymbol{x}}_l (ik_m) \underbrace{E_{0n} \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)]}_{E_n(\boldsymbol{r},t)} \qquad ; \text{ using Exercise - 3} \\
&= i \underbrace{\sum_{l,m,n=1}^{3} \mathcal{E}_{lmn}\ \hat{\boldsymbol{x}}_l k_m E_n(\boldsymbol{r}, t)}_{(\boldsymbol{k} \times \boldsymbol{E})} \\
\boldsymbol{\nabla} \times \boldsymbol{E}(\boldsymbol{r}, t) &= i\ \boldsymbol{k} \times \boldsymbol{E}(\boldsymbol{r}, t)
\end{aligned}$$

Similar expression for $\boldsymbol{B}(\boldsymbol{r}, t)$ can be obtained in the same way. Finally, we have

$$\boxed{\begin{aligned}
\boldsymbol{\nabla} \times \boldsymbol{E}(\boldsymbol{r}, t) &= i\ \boldsymbol{k} \times \boldsymbol{E}(\boldsymbol{r}, t) \\
\boldsymbol{\nabla} \times \boldsymbol{B}(\boldsymbol{r}, t) &= i\ \boldsymbol{k} \times \boldsymbol{B}(\boldsymbol{r}, t)
\end{aligned}} \tag{41}$$

$$\begin{aligned}
\frac{\partial}{\partial t} \boldsymbol{B}(\boldsymbol{r}, t) &= \frac{\partial}{\partial t} \boldsymbol{B}_0 \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)] \\
&= -i\omega \boldsymbol{B}_0 \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)] \\
\frac{\partial}{\partial t} \boldsymbol{B}(\boldsymbol{r}, t) &= -i\omega \boldsymbol{B}(\boldsymbol{r}, t)
\end{aligned} \tag{42}$$

Therefore,

$$\begin{aligned}
& \boldsymbol{\nabla} \times \boldsymbol{E}(\boldsymbol{r},t) = -\frac{\partial}{\partial t}\boldsymbol{B}(\boldsymbol{r},t) \\
\text{or,} \quad & i\,\boldsymbol{k} \times \boldsymbol{E}(\boldsymbol{r},t) = i\omega \boldsymbol{B}(\boldsymbol{r},t) \\
\text{or,} \quad & \boldsymbol{B}(\boldsymbol{r},t) = \frac{k}{\omega}\,\hat{\boldsymbol{k}} \times \boldsymbol{E}(\boldsymbol{r},t) = \frac{1}{c}\,\hat{\boldsymbol{k}} \times \boldsymbol{E}(\boldsymbol{r},t) \qquad ; \text{ since } \quad \omega = ck
\end{aligned} \tag{43}$$

Therefore, **relation between $\boldsymbol{E}(\boldsymbol{r},t)$ and $\boldsymbol{B}(\boldsymbol{r},t)$ in electromagnetic wave**,

$$\boxed{\boldsymbol{B}(\boldsymbol{r},t) = \frac{1}{c}\,\hat{\boldsymbol{k}} \times \boldsymbol{E}(\boldsymbol{r},t)} \tag{44}$$

So,

$$\hat{\boldsymbol{k}} \cdot \boldsymbol{B}(\boldsymbol{r},t) = \frac{1}{c}\hat{\boldsymbol{k}} \cdot (\hat{\boldsymbol{k}} \times \boldsymbol{E}(\boldsymbol{r},t)) = \frac{1}{c}\boldsymbol{E}(\boldsymbol{r},t) \cdot (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{k}}) = 0$$

In this section we have established **for electromagnetic wave**,

$$\boxed{\hat{\boldsymbol{k}} \cdot \boldsymbol{B}(\boldsymbol{r},t) = 0 \quad \text{and} \quad \hat{\boldsymbol{k}} \cdot \boldsymbol{E}(\boldsymbol{r},t) = 0 \quad \text{and} \quad \boldsymbol{E}(\boldsymbol{r},t) \cdot \boldsymbol{B}(\boldsymbol{r},t) = 0}$$

► That means, **in electromagnetic wave, electric field, magnetic field and wave vector are mutually perpendicular to each other**.

1.5 Potential formulation

From elementary idea of electrostatics it is observed that curl of electrostatic field $\boldsymbol{E}(\boldsymbol{r})$ vanishes,

$$\boldsymbol{\nabla} \times \boldsymbol{E}(\boldsymbol{r}) = 0$$

So that, electrostatic field can be derived from scalar potential $\phi(\boldsymbol{r})$ in the form of $\boldsymbol{E}(\boldsymbol{r}) = -\boldsymbol{\nabla}\phi(\boldsymbol{r})$ because *curl of gradient always vanishes*. Similar concept can be applied to the Maxwell's relations (8) to identify the potential functions from which electromagnetic fields $\boldsymbol{E}(\boldsymbol{r},t)$ and $\boldsymbol{B}(\boldsymbol{r},t)$ can be derived.

Maxwell's relation (8-ii) reads as,

$$\boldsymbol{\nabla} \cdot \boldsymbol{B}(\boldsymbol{r},t) = 0$$

Therefore, the $\boldsymbol{B}$ field can be derived from *vector potential* $\boldsymbol{A}(\boldsymbol{r},t)$ in the following way,

$$\boxed{\boldsymbol{B}(\boldsymbol{r},t) = \boldsymbol{\nabla} \times \boldsymbol{A}(\boldsymbol{r},t)} \tag{45}$$

because *divergence of curl always zero.*

Now, from (8-iii),

$$\begin{aligned}
& \boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} = -\frac{\partial}{\partial t}(\boldsymbol{\nabla} \times \boldsymbol{A}) \\
\text{or,} \quad & \boldsymbol{\nabla} \times \left(\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t}\right) = 0
\end{aligned}$$

Curl of gradient always zero, so, one can express,

$$\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t} = -\boldsymbol{\nabla}\phi$$

For time dependent phenomena, unlike to the static case,

$$\boxed{\boldsymbol{E}(\boldsymbol{r},t) = -\boldsymbol{\nabla}\phi(\boldsymbol{r},t) - \frac{\partial \boldsymbol{A}(\boldsymbol{r},t)}{\partial t}} \tag{46}$$

• So, the scalar potential $\phi(\boldsymbol{r},t)$ and vector potential $\boldsymbol{A}(\boldsymbol{r},t)$ determine the electromagnetic fields according to relations (45) and (46).

The fields do not correspond to unique potential functions. Potential functions are arbitrary upto an additive constant which does not alter the fields. Potentials can be transformed in proper way to new one such that fields remain invariant. Such transformations are known as **gauge transformation**.

Consider the transformations,

$$\begin{aligned} & \boldsymbol{A}(\boldsymbol{r},t) \rightarrow \boldsymbol{A'}(\boldsymbol{r},t) = \boldsymbol{A}(\boldsymbol{r},t) + \boldsymbol{\nabla}\xi(\boldsymbol{r},t) \\ \text{and} \qquad & \phi(\boldsymbol{r},t) \rightarrow \phi'(\boldsymbol{r},t) = \phi(\boldsymbol{r},t) - \frac{\partial}{\partial t}\xi(\boldsymbol{r},t) \end{aligned}$$

where $\xi(\boldsymbol{r},t)$ is any arbitrary scalar function.

Under this transformation,

$$\boldsymbol{B} \rightarrow \boldsymbol{B'} = \boldsymbol{\nabla}\times\boldsymbol{A'} = \boldsymbol{\nabla}\times(\boldsymbol{A}+\boldsymbol{\nabla}\xi) = \boldsymbol{\nabla}\times\boldsymbol{A} + \underbrace{\boldsymbol{\nabla}\times\boldsymbol{\nabla}\xi}_{0} = \boldsymbol{\nabla}\times\boldsymbol{A} = \boldsymbol{B}$$

because *curl of gradient always zero.*

The $\boldsymbol{E}$ field transforms as,

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{r},t) \rightarrow \boldsymbol{E'}(\boldsymbol{r},t) &= -\boldsymbol{\nabla}\phi'(\boldsymbol{r},t) - \frac{\partial \boldsymbol{A'}(\boldsymbol{r},t)}{\partial t} \\ &= -\boldsymbol{\nabla}\phi(\boldsymbol{r},t) - \frac{\partial \boldsymbol{A}(\boldsymbol{r},t)}{\partial t} + \boldsymbol{\nabla}\left(\frac{\partial \xi(\boldsymbol{r},t)}{\partial t}\right) - \frac{\partial}{\partial t}\boldsymbol{\nabla}\xi(\boldsymbol{r},t) \\ &= -\boldsymbol{\nabla}\phi(\boldsymbol{r},t) - \frac{\partial \boldsymbol{A}(\boldsymbol{r},t)}{\partial t} - \frac{\partial}{\partial t}\boldsymbol{\nabla}\xi(\boldsymbol{r},t) + \frac{\partial}{\partial t}\boldsymbol{\nabla}\xi(\boldsymbol{r},t) \\ &= -\boldsymbol{\nabla}\phi(\boldsymbol{r},t) - \frac{\partial \boldsymbol{A}(\boldsymbol{r},t)}{\partial t} \\ &= \boldsymbol{E}(\boldsymbol{r},t) \end{aligned}$$

► Under the following transformations of vector potential and scalar potential,

$$\boxed{\begin{aligned} & \boldsymbol{A}(\boldsymbol{r},t) \rightarrow \boldsymbol{A'}(\boldsymbol{r},t) = \boldsymbol{A}(\boldsymbol{r},t) + \boldsymbol{\nabla}\xi(\boldsymbol{r},t) \\ \text{and,} \qquad & \phi(\boldsymbol{r},t) \rightarrow \phi'(\boldsymbol{r},t) = \phi(\boldsymbol{r},t) - \frac{\partial}{\partial t}\xi(\boldsymbol{r},t) \end{aligned}} \tag{47}$$

both electric field $\boldsymbol{E}(\boldsymbol{r},t)$ and magnetic field $\boldsymbol{B}(\boldsymbol{r},t)$ remain unchanged. Hence the dynamics of a charged body moving in electromagnetic fields remain unaltered while the potentials transform in the prescribed way of (47).

► Maxwell's field equations can be re-expressed in term of potentials $\phi(\boldsymbol{r},t)$ and $\boldsymbol{A}(\boldsymbol{r},t)$.

Consider the Maxwell equation,

$$\boldsymbol{\nabla} \cdot \boldsymbol{E}(\boldsymbol{r}, t) = \frac{\rho(\boldsymbol{r}, t)}{\epsilon_0}$$

Using expression for $\boldsymbol{E}$ from (46)

$$\boldsymbol{\nabla} \cdot \left(-\boldsymbol{\nabla}\phi(\boldsymbol{r}, t) - \frac{\partial \boldsymbol{A}(\boldsymbol{r}, t)}{\partial t} \right) = \frac{\rho(\boldsymbol{r}, t)}{\epsilon_0}$$

$$\text{or,} \qquad \nabla^2 \phi + \frac{\partial}{\partial t}(\boldsymbol{\nabla} \cdot \boldsymbol{A}) = -\frac{\rho}{\epsilon_0} \tag{48}$$

Now consider another one Maxwell equation,

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$$

Using expression for $\boldsymbol{E}$ and $\boldsymbol{B}$ from (46) and (45)

$$\text{or,} \qquad \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = \mu_0 \boldsymbol{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left(-\boldsymbol{\nabla}\phi(\boldsymbol{r}, t) - \frac{\partial \boldsymbol{A}(\boldsymbol{r}, t)}{\partial t} \right)$$

$$\text{or,} \qquad \nabla^2 \boldsymbol{A} - \boldsymbol{\nabla} \left(\boldsymbol{\nabla} \cdot \boldsymbol{A} + \mu_0 \epsilon_0 \frac{\partial \phi}{\partial t} \right) - \mu_0 \epsilon_0 \frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\mu_0 \boldsymbol{J} \tag{49}$$

In **Lorentz gauge**,

$$\boxed{\boldsymbol{\nabla} \cdot \boldsymbol{A} + \mu_0 \epsilon_0 \frac{\partial \phi}{\partial t} = 0} \tag{50}$$

the equation (49) becomes,

$$\boxed{\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \boldsymbol{A}(\boldsymbol{r}, t) = -\mu_0 \boldsymbol{J}(\boldsymbol{r}, t)} \qquad ; \text{ since,} \quad c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \tag{51}$$

and equation (48) becomes

$$\boxed{\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \phi(\boldsymbol{r}, t) = -\frac{\rho(\boldsymbol{r}, t)}{\epsilon_0}} \qquad ; \text{ since,} \quad c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \tag{52}$$

Both scalar potential and the magnetic vector potential exhibit wave equation with source term.

1.5.1 Lagrangian of charged particle moving in electromagnetic field

Force experienced by a point charge q moving in electromagnetic field (given by $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{B}(\boldsymbol{r}, t)$) is described by **Lorentz force**,

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \tag{53}$$

From, section-1.5,

$$\boldsymbol{B}(\boldsymbol{r}, t) = \boldsymbol{\nabla} \times \boldsymbol{A}(\boldsymbol{r}, t)$$

$$\text{and,} \qquad \boldsymbol{E}(\boldsymbol{r}, t) = -\boldsymbol{\nabla}\phi(\boldsymbol{r}, t) - \frac{\partial \boldsymbol{A}(\boldsymbol{r}, t)}{\partial t}$$

Therefore, one can express Lorentz force in terms of potential functions,

$$\begin{aligned} \boldsymbol{F} &= q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \\ &= q \left[-\boldsymbol{\nabla}\phi(\boldsymbol{r}, t) - \frac{\partial \boldsymbol{A}(\boldsymbol{r}, t)}{\partial t} + \boldsymbol{v} \times (\boldsymbol{\nabla} \times \boldsymbol{A}(\boldsymbol{r}, t)) \right] \end{aligned} \tag{54}$$

Now use the vector identity,

$$
\begin{aligned}
& \boldsymbol{\nabla}(\boldsymbol{v}\cdot\boldsymbol{A}) = \boldsymbol{v}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) + \boldsymbol{A}\times(\boldsymbol{\nabla}\times\boldsymbol{v}) + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{A} + (\boldsymbol{A}\cdot\boldsymbol{\nabla})\boldsymbol{v} \\
& \text{generalized velocity } \boldsymbol{v} \text{ is independent of generalized position} \\
& \text{so, space derivative of } \boldsymbol{v} \text{ vanishes} \\
\Rightarrow \quad & \boldsymbol{\nabla}(\boldsymbol{v}\cdot\boldsymbol{A}) = \boldsymbol{v}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{A} \\
\Rightarrow \quad & \boldsymbol{v}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{v}\cdot\boldsymbol{A}) - (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{A}
\end{aligned}
$$

So, equation (54) becomes,

$$
\boldsymbol{F} = q\left[-\boldsymbol{\nabla}\phi(\boldsymbol{r},t) - \frac{\partial \boldsymbol{A}(\boldsymbol{r},t)}{\partial t} + \boldsymbol{\nabla}(\boldsymbol{v}\cdot\boldsymbol{A}) - (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{A}\right] \tag{55}
$$

Now, consider the Cartesian coordinate axes $x_1,\ x_2,\ x_3$ (which are identical to $x,\ y,\ z$).

$$
\begin{aligned}
& \boldsymbol{A} = \boldsymbol{A}(\boldsymbol{r},t) = \boldsymbol{A}(x_1,x_2,x_3,t) \\
\Rightarrow \quad & d\boldsymbol{A} = \sum_{i=1}^{3} \frac{\partial \boldsymbol{A}}{\partial x_i} dx_i + \frac{\partial \boldsymbol{A}}{\partial t} dt \\
\Rightarrow \quad & \frac{d\boldsymbol{A}}{dt} = \sum_{i=1}^{3} v_i \frac{\partial \boldsymbol{A}}{\partial x_i} + \frac{\partial \boldsymbol{A}}{\partial t} \\
\Rightarrow \quad & \frac{d\boldsymbol{A}}{dt} = (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{A} + \frac{\partial \boldsymbol{A}}{\partial t} \\
\Rightarrow \quad & (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{A} = \frac{d\boldsymbol{A}}{dt} - \frac{\partial \boldsymbol{A}}{\partial t}
\end{aligned}
$$

Now, substitute this expression of $(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{A}$ into equation (55) and get,

$$
\boxed{\boldsymbol{F} = -\boldsymbol{\nabla}(q\phi - q\boldsymbol{v}\cdot\boldsymbol{A}) - \frac{d}{dt}(q\boldsymbol{A})} \tag{56}
$$

This is the expression for electromagnetic force (in terms of potentials $\phi(\boldsymbol{r},t)$ and $\boldsymbol{A}(\boldsymbol{r},t)$) on a moving charged particle.

The equation of motion for this force,

$$
\begin{aligned}
& m\frac{d\boldsymbol{v}}{dt} - \boldsymbol{F} = 0 \\
\text{or,} \quad & m\frac{d\boldsymbol{v}}{dt} + \boldsymbol{\nabla}(q\phi - q\boldsymbol{v}\cdot\boldsymbol{A}) + \frac{d}{dt}(q\boldsymbol{A}) = 0
\end{aligned}
$$

For the j-th Cartesian component, equation of motion,

$$
m\frac{dv_j}{dt} + \frac{\partial}{\partial x_j}(q\phi - q\boldsymbol{v}\cdot\boldsymbol{A}) + \frac{d}{dt}(qA_j) = 0 \tag{57}
$$

The Lagrangian $L(\boldsymbol{r},\boldsymbol{v},t)$ for the charged particle moving in electromagnetic field is to be identified such that the Euler-Lagrange equation of motion,

$$
\frac{d}{dt}\left(\frac{\partial L}{\partial v_j}\right) - \frac{\partial L}{\partial x_j} = 0
$$

gives us the identical equation of motion expressed by (57).

To get the explicit expression for Lagrangian, we will rearrange the equation of motion (57) in the form of Euler-Lagrange equation.

To rearrange the first term of (57) consider,

$$\frac{\partial v_i}{\partial v_j} = \delta_{ij} \qquad \text{and} \qquad v^2 = \sum_{i=1}^{3} v_i v_i$$

Therefore,

$$m\frac{dv_j}{dt} = \frac{d}{dt}\frac{\partial}{\partial v_j}\left(\frac{1}{2}m\sum_{i=1}^{3} v_i v_i\right) = \frac{d}{dt}\frac{\partial}{\partial v_j}\left(\frac{1}{2}mv^2\right) = \frac{d}{dt}\frac{\partial T}{\partial v_j} \tag{58}$$

The kinetic energy T has no explicit space dependency, therefore,

$$\frac{\partial T}{\partial x_j} = 0 \qquad \text{; for any of } j = 1, 2, 3$$

So, second term of (57),

$$\frac{\partial}{\partial x_j}(q\phi - q\boldsymbol{v}\cdot\boldsymbol{A}) = -\frac{\partial}{\partial x_j}[T - q(\phi - \boldsymbol{v}\cdot\boldsymbol{A})] \tag{59}$$

The scalar potential $\phi(\boldsymbol{r}, t)$ and the vector potential $\boldsymbol{A}(\boldsymbol{r}, t)$ does not have any velocity dependency, therefore,

$$\frac{\partial \phi}{\partial v_j} = 0 = \frac{\partial \boldsymbol{A}}{\partial v_j} \qquad \text{; for any of } j = 1, 2, 3$$

and

$$\sum_{i=1}^{3} v_i A_i = \boldsymbol{v}\cdot\boldsymbol{A}$$

So, the third term of (57),

$$\frac{d}{dt}(qA_j) = -\frac{d}{dt}\frac{\partial}{\partial v_j}\left(q\phi - q\sum_{i=1}^{3} v_i A_i\right) = -\frac{d}{dt}\frac{\partial}{\partial v_j}[q(\phi - \boldsymbol{v}\cdot\boldsymbol{A})] \tag{60}$$

Now substitute several terms from (58), (59) and (60) into (57),

$$\begin{aligned} &\frac{d}{dt}\frac{\partial T}{\partial v_j} - \frac{\partial}{\partial x_j}[T - q(\phi - \boldsymbol{v}\cdot\boldsymbol{A})] - \frac{d}{dt}\frac{\partial}{\partial v_j}[q(\phi - \boldsymbol{v}\cdot\boldsymbol{A})] = 0 \\ \text{or,} \quad &\frac{d}{dt}\frac{\partial}{\partial v_j}[T - q(\phi - \boldsymbol{v}\cdot\boldsymbol{A})] - \frac{\partial}{\partial x_j}[T - q(\phi - \boldsymbol{v}\cdot\boldsymbol{A})] = 0 \\ \text{or,} \quad &\frac{d}{dt}\left(\frac{\partial L}{\partial v_j}\right) - \frac{\partial L}{\partial x_j} = 0 \qquad \text{; the Euler-Lagrange equation} \end{aligned} \tag{61}$$

Here, we able to find out the appropriate Lagrangian from the equation of motion as,

$$\boxed{L = T - q(\phi - \boldsymbol{v}\cdot\boldsymbol{A})} \tag{62}$$

This is the **Lagrangian for charged particle moving in electromagnetic field**.
The **potential U is velocity dependent**

$$U = q(\phi - \boldsymbol{v}\cdot\boldsymbol{A})$$

1.5.2 Transformation of Lagrangian due to gauge transformation

It has been discussed that under the transformation of $\phi(\boldsymbol{r},t)$ and $\boldsymbol{A}(\boldsymbol{r},t)$ according to (47), the electromagnetic fields $\boldsymbol{E}(\boldsymbol{r},t)$ and $\boldsymbol{B}(\boldsymbol{r},t)$ remain unaltered. So, the Lorentz force expression (53) also to be kept unchanged. Let's check what happens to the Lagrangian due to the transformation of potentials,

$$\begin{aligned} &\boldsymbol{A}(\boldsymbol{r},t) \to \boldsymbol{A'}(\boldsymbol{r},t) = \boldsymbol{A}(\boldsymbol{r},t) + \boldsymbol{\nabla}\xi(\boldsymbol{r},t) \\ \text{and,}\qquad &\phi(\boldsymbol{r},t) \to \phi'(\boldsymbol{r},t) = \phi(\boldsymbol{r},t) - \frac{\partial}{\partial t}\xi(\boldsymbol{r},t) \end{aligned}$$

The transformed Lagrangian,

$$\begin{aligned} L'(\boldsymbol{r},\boldsymbol{v},t) &= T - q(\phi' - \boldsymbol{v}\cdot\boldsymbol{A'}) \\ &= T - q\left(\phi - \frac{\partial}{\partial t}\xi(\boldsymbol{r},t) - \boldsymbol{v}\cdot\boldsymbol{A} - \boldsymbol{v}\cdot\boldsymbol{\nabla}\xi(\boldsymbol{r},t)\right) \\ &= \underbrace{T - q(\phi - \boldsymbol{v}\cdot\boldsymbol{A})}_{L} - \left(\boldsymbol{v}\cdot\boldsymbol{\nabla}\xi(\boldsymbol{r},t) + \frac{\partial}{\partial t}\xi(\boldsymbol{r},t)\right) \\ &= L - \left(\boldsymbol{v}\cdot\boldsymbol{\nabla}\xi(\boldsymbol{r},t) + \frac{\partial}{\partial t}\xi(\boldsymbol{r},t)\right) \end{aligned} \tag{63}$$

Now,

$$\begin{aligned} &\xi = \xi(\boldsymbol{r},t) = \xi(x_1,x_2,x_3,t) \\ \Rightarrow\qquad &d\xi(\boldsymbol{r},t) = \sum_{i=1}^{3}\frac{\partial\xi}{\partial x_i}dx_i + \frac{\partial\xi}{\partial t}dt \\ \Rightarrow\qquad &\frac{d\xi}{dt} = \sum_{i=1}^{3}v_i\frac{\partial\xi}{\partial x_i} + \frac{\partial\xi}{\partial t} = \boldsymbol{v}\cdot\boldsymbol{\nabla}\xi(\boldsymbol{r},t) + \frac{\partial}{\partial t}\xi(\boldsymbol{r},t) \end{aligned}$$

So, from the last expression of (63),

$$L' = L - \frac{d\xi(\boldsymbol{r},t)}{dt}$$

The Lagrangian is transformed such that it is arbitrary upto an additive factor which is total time derivative of a function of space and time only (here $\xi(\boldsymbol{r},t)$). From the idea of Lagrangian dynamics, it is known that such an addition does not affect the Euler-Lagrange equation of motion. Therefore, gauge transformations of potentials do not alter the force acting on charged particle and hence the equation of motion remain unchanged.

1.5.3 Hamiltonian

The Lagrangian for charged particle moving in electromagnetic fields is given by relation (62) as,

$$\begin{aligned} L &= T - q[\phi(\boldsymbol{r},t) - \boldsymbol{v}\cdot\boldsymbol{A}(\boldsymbol{r},t)] \\ &= \underbrace{\frac{1}{2}m\sum_{i=1}^{3}v_iv_i}_{T} - q\phi(\boldsymbol{r},t) + q\underbrace{\sum_{i=1}^{3}v_iA_i(\boldsymbol{r},t)}_{\boldsymbol{v}\cdot\boldsymbol{A}} \end{aligned}$$

Now the **$j-$th component of generalized momentum**,

$$\begin{aligned} p_j &= \frac{\partial L}{\partial v_j} \\ &= \frac{\partial}{\partial v_j}\left(\frac{1}{2}m\sum_{i=1}^{3} v_i v_i - q\phi(\boldsymbol{r},t) + q\sum_{i=1}^{3} v_i A_i(\boldsymbol{r},t)\right) \\ &= m\sum_{i=1}^{3} v_i\delta_{ij} + q\sum_{i=1}^{3} A_i(\boldsymbol{r},t)\delta_{ij} \qquad ; \text{ since } \quad \frac{\partial v_i}{\partial v_j} = \delta_{ij} \\ p_j &= mv_j + qA_j(\boldsymbol{r},t) \end{aligned}$$

The **generalized momentum**,

$$\boxed{\boldsymbol{p} = m\boldsymbol{v} + q\boldsymbol{A}(\boldsymbol{r},t)} \tag{64}$$

where,

$m\boldsymbol{v} \rightarrow$ mechanical momentum due to motion of charged particle.

$q\boldsymbol{A}(\boldsymbol{r},t) \rightarrow$ field momentum carried by electromagnetic fields.

The **Hamiltonian** can be formulated as,

$$\begin{aligned} H &= \sum_{i=1}^{3} p_i v_i - L \qquad ; \text{ Basic definition} \\ &= \boldsymbol{p}\cdot\boldsymbol{v} - T + q\phi - q\boldsymbol{v}\cdot\boldsymbol{A} \qquad ; \text{ Lagrangian expression from (62)} \\ &\text{Now put from equation (64),} \quad \boldsymbol{v} = \frac{1}{m}(\boldsymbol{p} - q\boldsymbol{A}) \\ H &= \frac{1}{m}\boldsymbol{p}\cdot(\boldsymbol{p} - q\boldsymbol{A}) - \frac{1}{2m}(\boldsymbol{p} - q\boldsymbol{A})^2 + q\phi - \frac{q}{m}(\boldsymbol{p} - q\boldsymbol{A})\cdot\boldsymbol{A} \\ &= \frac{1}{2m}(\boldsymbol{p} - q\boldsymbol{A})^2 + q\phi \end{aligned}$$

So, the **Hamiltonian of charged particles moving in electromagnetic field**,

$$\boxed{H(\boldsymbol{r},\boldsymbol{p},t) = \frac{1}{2m}[\boldsymbol{p} - q\boldsymbol{A}(\boldsymbol{r},t)]^2 + q\phi(\boldsymbol{r},t)} \tag{65}$$

2 Energy conservation : Poynting's theorem

Energy conservation is one of the fundamental laws of physics. If total energy of a system is increased by some amount that means the same amount of energy is flowed into the system and on the other hand when some amount of energy is flowed out form system then same amount of energy of the system will be decreased. The energy conservation must leads to a continuity equation in the form of,

$$\frac{\partial u}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{S} = 0$$

where, u is the energy density, i.e; energy per unit volume. $\boldsymbol{S}$ is the energy flux density, i.e; energy flowing per unit time across unit surface along the direction of energy flow.

Following, we discuss the energy conservation phenomena in electrodynamics and also going to develop the continuity equation associated with the energy conservation.

Consider a region of volume V enclosed by surface $\mathcal{R}$, in which there are some charges of charge density $\rho(\boldsymbol{r},t)$ and currents of current density $\boldsymbol{J}(\boldsymbol{r},t)$; both charge density and current density are space-time dependent variable. Furthermore, there will be electric field $\boldsymbol{E}(\boldsymbol{r},t)$ and magnetic field $\boldsymbol{B}(\boldsymbol{r},t)$. This electromagnetic field originates from the charges and currents present within the region plus some external sources can give rise to fields within the region of interest $\mathcal{R}$.

Suppose, small volume element $d\tau$ within the region is moving with velocity $\boldsymbol{v}$.

Now, total charge contained in the volume element is $dq = \rho(\boldsymbol{r},t)d\tau$, then Lorentz force acting on the small element is,

$$\boldsymbol{F} = dq(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}) = (\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})\rho(\boldsymbol{r},t)d\tau$$

If within small time interval dt, the volume element is displaced by $d\boldsymbol{l} = \boldsymbol{v}dt$, then work done on the whole system is,

$$dW = \int_V \boldsymbol{F}\cdot d\boldsymbol{l}$$

$$\text{or,} \qquad dW = \int_V \underbrace{\rho(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})d\tau}_{\boldsymbol{F}}\cdot\underbrace{\boldsymbol{v}dt}_{d\boldsymbol{l}}$$

$$\text{or,} \qquad dW = \int_V \rho\boldsymbol{E}\cdot\boldsymbol{v}\;d\tau\;dt \qquad ;\ \text{since,}\quad (\boldsymbol{v}\times\boldsymbol{B})\cdot\boldsymbol{v} = 0$$

$$\text{or,} \qquad dW = \int_V \boldsymbol{E}\cdot\boldsymbol{J}\;d\tau\;dt \qquad ;\ \text{since,}\quad \boldsymbol{J} = \rho\boldsymbol{v}$$

Now put $\boldsymbol{J}$ form Maxwell equation : $\boldsymbol{\nabla}\times\boldsymbol{B} = \mu_0\boldsymbol{J} + \mu_0\epsilon_0\dfrac{\partial\boldsymbol{E}}{\partial t}$

$$\text{or,} \qquad \frac{dW}{dt} = \int_V \boldsymbol{E}\cdot\left(\frac{1}{\mu_0}\boldsymbol{\nabla}\times\boldsymbol{B} - \epsilon_0\frac{\partial\boldsymbol{E}}{\partial t}\right)\;d\tau$$

$$\text{or,} \qquad \frac{dW}{dt} = \frac{1}{\mu_0}\int_V \boldsymbol{E}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B})\;d\tau - \frac{d}{dt}\int_V \frac{\epsilon_0}{2}E^2\;d\tau \qquad ;\ \text{since,}\quad E^2 = \boldsymbol{E}\cdot\boldsymbol{E}$$

Now use the vector identity : $\boldsymbol{\nabla}\cdot(\boldsymbol{E}\times\boldsymbol{B}) = \boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{E}) - \boldsymbol{E}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B})$

$$\text{or,} \qquad \frac{dW}{dt} = \frac{1}{\mu_0}\int_V \boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{E})\;d\tau - \frac{1}{\mu_0}\int_V \boldsymbol{\nabla}\cdot(\boldsymbol{E}\times\boldsymbol{B})\;\;d\tau - \frac{d}{dt}\int_V \frac{\epsilon_0}{2}E^2\;d\tau$$

Now use Maxwell equation : $\boldsymbol{\nabla}\times\boldsymbol{E} = -\dfrac{\partial\boldsymbol{B}}{\partial t}$

$$\text{or,} \qquad \frac{dW}{dt} = -\frac{1}{\mu_0}\int_V \boldsymbol{B}\cdot\frac{\partial\boldsymbol{B}}{\partial t}\;d\tau - \frac{1}{\mu_0}\int_V \boldsymbol{\nabla}\cdot(\boldsymbol{E}\times\boldsymbol{B})\;\;d\tau - \frac{d}{dt}\int_V \frac{\epsilon_0}{2}E^2\;d\tau$$

$$\text{or,} \qquad \frac{dW}{dt} = -\frac{d}{dt}\int_V \left(\frac{\epsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2\right)d\tau - \frac{1}{\mu_0}\int_V \boldsymbol{\nabla}\cdot(\boldsymbol{E}\times\boldsymbol{B})d\tau \tag{66}$$

The work done on the system will be stored as mechanical energy. If $u_{mech}(\boldsymbol{r},t)$ is the mechanical energy density of the system of charges, then total mechanical energy will be equal to,

$$\int_V u_{mech}(\boldsymbol{r},t)\;d\tau$$

Rate of work done must be equal to the rate of change of mechanical energy, therefore,

$$\frac{dW}{dt} = \frac{d}{dt}\int_V u_{mech}(\boldsymbol{r},t)\;d\tau$$

So, the equation (66) becomes,

$$\int_V \left[\frac{\partial}{\partial t}\left(u_{mech} + \frac{\epsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2\right) + \boldsymbol{\nabla}\cdot\left(\frac{1}{\mu_0}\boldsymbol{E}\times\boldsymbol{B}\right)\right] d\tau = 0 \tag{67}$$

This is true for any arbitrary volume, so, we have continuity equation in the form of,

$$\boxed{\begin{array}{l} \textbf{Continuity equation :} \quad \dfrac{\partial u}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{S} = 0 \\ \text{where, } \textbf{Poynting vector :} \;\; \boldsymbol{S} = \dfrac{1}{\mu_0}(\boldsymbol{E}\times\boldsymbol{B}) \\ \text{where, } \quad u = u_{mech} + u_{em} \\ \text{Electrmagnetic energy density : } \; u_{em} = \dfrac{1}{2}\left(\epsilon_0 E^2 + \dfrac{1}{\mu_0}B^2\right) \end{array}} \tag{68}$$

► **Poynting's Theorem :** *The work on the charges by the electromagnetic force is equal the to decrease in energy stored in the field, less the energy that flowed out through the surface.*[4]

Exercise 4. Suppose $Ae^{i\alpha x} + Be^{i\beta x} = Ce^{i\gamma x}$ for some nonzero constants $A,\ B,\ C,\ \alpha,\ \beta,\ \gamma$ and for all x. Then show that $\alpha = \beta = \gamma$ and $A + B = C$

Solution :

The relation is valid for all x, so, at $x = 0$,

$$\boxed{A + B = C}$$

Differentiating the relation,

$$i\alpha Ae^{i\alpha x} + i\beta Be^{i\beta x} = i\gamma Ce^{i\gamma x}$$

$$\text{at } x = 0 : \quad \boxed{\alpha A + \beta B = \gamma C}$$

Differentiating again,

$$-\alpha^2 Ae^{i\alpha x} - \beta^2 Be^{i\beta x} = -\gamma^2 Ce^{i\gamma x}$$

At $x = 0$,

$$\begin{aligned} & \alpha^2 A + \beta^2 B = \gamma^2 C \\ \text{or,} \quad & C(\alpha^2 A + \beta^2 B) = (\gamma C)^2 \qquad \text{; multiplying both side by } C \\ \text{or,} \quad & (A+B)(\alpha^2 A + \beta^2 B) = (\alpha A + \beta B)^2 \qquad \text{; using expression for } C \text{ and } \gamma C \\ \text{or,} \quad & (\alpha A)^2 + (\beta B)^2 + (\alpha^2 + \beta^2)AB - (\alpha A + \beta B)^2 = 0 \\ \text{or,} \quad & (\alpha^2 + \beta^2 - 2\alpha\beta)AB = 0 \\ \text{or,} \quad & (\alpha - \beta)^2 AB = 0 \\ \text{or,} \quad & \boxed{\alpha = \beta} \qquad \text{; since } A,\ B\ \neq 0 \end{aligned}$$

We have,

$$\begin{aligned} & \alpha A + \beta B = \gamma C \\ \text{or,} \quad & \alpha(A + B) = \gamma C \qquad \text{; since } \alpha = \beta \\ \text{or,} \quad & \boxed{\alpha = \gamma} \qquad \text{; since } A + B = C \end{aligned} \tag{69}$$

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[4]Taken from D. J. Griffiths, Introduction to Electrodynamics, 3rd Eds., Prentice Hall (1999).