Partial Differential Equation

INTRODUCTION

Many of the problems of mathematical physics involve the solution of partial differential equations. The same partial differential equation may apply to a variety of physical problems; thus the mathematical methods which you will learn in this chapter apply to many more problems than those we shall discuss in the illustrative examples. Let us outline the partial differential equations we shall consider, and the kinds of physical problems which lead to each of them.

(1.1) **Laplace's equation** $\nabla^2 u = 0$

The function u may be the gravitational potential in a region containing no mass, the electrostatic potential in a charge-free region, the steady-state temperature (that is, temperature not changing with time) in a region containing no sources of heat, or the velocity potential for an incompressible fluid with no vortices and no sources or sinks.

(1.2) **Poisson's equation** $\nabla^2 u = f(x, y, z)$

The function u may represent the same physical quantities listed for Laplace's equation, but in a region containing mass, electric charge, or sources of heat or fluid, respectively, for the various cases. The function $f(x, y, z)$ is called the source density; for example, in electricity it is proportional to the density of the electric charge.

(1.3) **The diffusion or heat flow equation** $\nabla^2 u = \dfrac{1}{\alpha^2}\dfrac{\partial u}{\partial t}$

Here u may be the non-steady-state temperature (that is, temperature varying with time) in a region with no heat sources; or it may be the concentration of a diffusing substance (for example, a chemical, or particles such as neutrons). The quantity α^2 is a constant known as the diffusivity.

(1.4) **Wave equation** $\nabla^2 u = \dfrac{1}{v^2}\dfrac{\partial^2 u}{\partial t^2}$

Here u may represent the displacement from equilibrium of a vibrating string or membrane or (in acoustics) of the vibrating medium (gas, liquid, or solid); in electricity u may be the current or potential along a transmission line; or u may be a component of **E** or **B** in an electromagnetic wave (light, radio waves, etc.). The quantity v is the speed of propagation of the waves; for example, for light in a vacuum it is c, the speed of light, and for sound waves it is the speed at which sound travels in the medium under consideration. The operator $\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ is called the d'Alembertian.

(1.5) **Helmholtz equation** $\nabla^2 F + k^2 F = 0$

As you will see later, the function F here represents the space part (that is, the time-independent part) of the solution of either the diffusion or the wave equation.

(1.6) **Schrödinger equation** $-\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi = i\hbar\frac{\partial}{\partial t}\Psi$

This is the wave equation of quantum mechanics. In this equation, $\hbar$ is Planck's constant divided by 2π, m is the mass of a particle, $i = \sqrt{-1}$, and V is the potential energy of the particle. The wave function Ψ is complex, and its absolute square is proportional to the position probability of the particle.

We shall be principally concerned with the solution of these equations rather than their derivation. If you like, you could say that it is true experimentally that the physical quantities mentioned above satisfy the given equations. However, it is also true that the equations can be derived from somewhat simpler experimental assumptions. Let us indicate briefly an example of how this can be done. In Chapter 6, Sections 10 and 11, we considered the flow of a fluid. We showed (Chapter 6, Problem 10.15) that $\boldsymbol{\nabla}\cdot\mathbf{v} = 0$ for an incompressible fluid in a region containing no sources or sinks. If it is also true that there are no vortices (that is, the flow is irrotational), then curl $\mathbf{v} = 0$, and $\mathbf{v}$ can be written as the gradient of a scalar function: $\mathbf{v} = \boldsymbol{\nabla}u$. Combining these two equations, we have $\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}u = \nabla^2 u = 0$. The function u is called the velocity potential and we see that (under the given conditions) it satisfies Laplace's equation as we claimed. A few more examples of such derivations are outlined in the problems.

In the following sections, we shall consider a number of physical problems to illustrate the very useful method of solving partial differential equations known as *separation of variables* (no relation to the same term used in ordinary differential equations, Chapter 8). In Sections 2 to 4, we consider problems in rectangular coordinates leading to Fourier series solutions—problems similar to those solved by Fourier. In later sections, we consider use of other coordinate systems (cylindrical, spherical) leading to solutions using Legendre or Bessel series.

21.1 Separation of variables: the general method

Suppose we seek a solution $u(x, y, z, t)$ to some PDE (expressed in Cartesian coordinates). Let us attempt to obtain one that has the product form[§]

$$u(x, y, z, t) = X(x)Y(y)Z(z)T(t). \tag{21.1}$$

A solution that has this form is said to be *separable* in x, y, z and t, and seeking solutions of this form is called the method of *separation of variables*.

As simple examples we may observe that, of the functions

$$\text{(i) } xyz^2 \sin bt, \qquad \text{(ii) } xy + zt, \qquad \text{(iii) } (x^2 + y^2)z \cos \omega t,$$

(i) is completely separable, (ii) is inseparable in that no single variable can be separated out from it and written as a multiplicative factor, whilst (iii) is separable in z and t but not in x and y.

When seeking PDE solutions of the form (21.1), we are requiring not that there is no connection at all between the functions X, Y, Z and T (for example, certain parameters may appear in two or more of them), but only that X does not depend upon y, z, t, that Y does not depend on x, z, t, and so on.

For a general PDE it is likely that a separable solution is impossible, but certainly some common and important equations do have useful solutions of this form, and we will illustrate the method of solution by studying the three-dimensional wave equation

$$\nabla^2 u(\mathbf{r}) = \frac{1}{c^2}\frac{\partial^2 u(\mathbf{r})}{\partial t^2}. \tag{21.2}$$

We will work in Cartesian coordinates for the present and assume a solution of the form (21.1); the solutions in alternative coordinate systems, e.g. spherical or cylindrical polars, are considered in section 21.3. Expressed in Cartesian coordinates (21.2) takes the form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}; \tag{21.3}$$

substituting (21.1) gives

$$\frac{d^2 X}{dx^2} YZT + X\frac{d^2 Y}{dy^2} ZT + XY\frac{d^2 Z}{dz^2} T = \frac{1}{c^2} XYZ\frac{d^2 T}{dt^2},$$

which can also be written as

$$X''YZT + XY''ZT + XYZ''T = \frac{1}{c^2}XYZT'', \tag{21.4}$$

where in each case the primes refer to the *ordinary* derivative with respect to the independent variable upon which the function depends. This emphasises the fact that each of the functions X, Y, Z and T has only one independent variable and thus its only derivative is its total derivative. For the same reason, in each term in (21.4) three of the four functions are unaltered by the partial differentiation and behave exactly as constant multipliers.

If we now divide (21.4) throughout by $u = XYZT$ we obtain

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = \frac{1}{c^2}\frac{T''}{T}. \tag{21.5}$$

This form shows the particular characteristic that is the basis of the method of separation of variables, namely that of the four terms the first is a function of x only, the second of y only, the third of z only and the RHS a function of t only and yet there is an equation connecting them. This can only be so for all x, y, z and t if *each* of the terms does not in fact, despite appearances, depend upon the corresponding independent variable but *is equal to a constant*, the four constants being such that (21.5) is satisfied.

Since there is only one equation to be satisfied and four constants involved, there is considerable freedom in the values they may take. For the purposes of our illustrative example let us make the choice of $-l^2$, $-m^2$, $-n^2$, for the first three constants. The constant associated with $c^{-2}T''/T$ must then have the value $-\mu^2 = -(l^2 + m^2 + n^2)$.

Having recognised that each term of (21.5) is individually equal to a constant (or parameter), we can now replace (21.5) by four separate ordinary differential equations (ODEs):

$$\frac{X''}{X} = -l^2, \qquad \frac{Y''}{Y} = -m^2, \qquad \frac{Z''}{Z} = -n^2, \qquad \frac{1}{c^2}\frac{T''}{T} = -\mu^2. \tag{21.6}$$

The important point to notice is not the simplicity of the equations (21.6) (the corresponding ones for a general PDE are usually far from simple) but that, by the device of assuming a separable solution, a *partial* differential equation (21.3), containing derivatives with respect to the four independent variables all in one equation, has been reduced to four *separate ordinary* differential equations (21.6). The ordinary equations are connected through four constant parameters that satisfy an algebraic relation. These constants are called *separation constants.*

The general solutions of the equations (21.6) can be deduced straightforwardly and are

$$\begin{aligned} X(x) &= A\exp(ilx) + B\exp(-ilx), \\ Y(y) &= C\exp(imy) + D\exp(-imy), \\ Z(z) &= E\exp(inz) + F\exp(-inz), \\ T(t) &= G\exp(ic\mu t) + H\exp(-ic\mu t), \end{aligned} \tag{21.7}$$

where A, $B, \ldots, H$ are constants, which may be determined if boundary condtions are imposed on the solution. Depending on the geometry of the problem and any boundary conditions, it is sometimes more appropriate to write the solutions (21.7) in the alternative form

$$\begin{aligned} X(x) &= A'\cos lx + B'\sin lx, \\ Y(y) &= C'\cos my + D'\sin my, \\ Z(z) &= E'\cos nz + F'\sin nz, \\ T(t) &= G'\cos(c\mu t) + H'\sin(c\mu t), \end{aligned} \tag{21.8}$$

for some different set of constants $A', B', \ldots, H'$. Clearly the choice of how best to represent the solution depends on the problem being considered.

As an example, suppose that we take as particular solutions the four functions

$$X(x) = \exp(ilx), \qquad Y(y) = \exp(imy),$$

$$Z(z) = \exp(inz), \qquad T(t) = \exp(-ic\mu t).$$

This gives a particular solution of the original PDE (21.3)

$$\begin{aligned} u(x, y, z, t) &= \exp(ilx)\exp(imy)\exp(inz)\exp(-ic\mu t) \\ &= \exp[i(lx + my + nz - c\mu t)], \end{aligned}$$

►*Use the method of separation of variables to obtain for the one-dimensional diffusion equation*

$$\kappa\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \tag{21.9}$$

a solution that tends to zero as $t \to \infty$ for all x.

Here we have only two independent variables x and t and we therefore assume a solution of the form

$$u(x, t) = X(x)T(t).$$

Substituting this expression into (21.9) and dividing through by $u = XT$ (and also by κ) we obtain

$$\frac{X''}{X} = \frac{T'}{\kappa T}.$$

Now, arguing exactly as above that the LHS is a function of x only and the RHS is a function of t only, we conclude that each side must equal a constant, which, anticipating the result and noting the imposed boundary condition, we will take as $-\lambda^2$. This gives us two ordinary equations,

$$X'' + \lambda^2 X = 0, \tag{21.10}$$

$$T' + \lambda^2 \kappa T = 0, \tag{21.11}$$

which have the solutions

$$X(x) = A\cos\lambda x + B\sin\lambda x,$$

$$T(t) = C\exp(-\lambda^2\kappa t).$$

Combining these to give the assumed solution $u = XT$ yields (absorbing the constant C into A and B)

$$u(x,t) = (A\cos\lambda x + B\sin\lambda x)\exp(-\lambda^2\kappa t). \tag{21.12}$$

In order to satisfy the boundary condition $u \to 0$ as $t \to \infty$, $\lambda^2\kappa$ must be > 0. Since κ is real and > 0, this implies that λ is a real non-zero number and that the solution is sinusoidal in x and is not a disguised hyperbolic function; this was our reason for choosing the separation constant as $-\lambda^2$. ◄

►*Use the method of separation of variables to obtain a solution for the two-dimensional Laplace equation,*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{21.13}$$

If we assume a solution of the form $u(x,y) = X(x)Y(y)$ then, following the above method, and taking the separation constant as λ^2, we find

$$X'' = \lambda^2 X, \qquad Y'' = -\lambda^2 Y.$$

Taking λ^2 as > 0, the general solution becomes

$$u(x,y) = (A\cosh\lambda x + B\sinh\lambda x)(C\cos\lambda y + D\sin\lambda y). \tag{21.14}$$

An alternative form, in which the exponentials are written explicitly, may be useful for other geometries or boundary conditions:

$$u(x,y) = [A\exp\lambda x + B\exp(-\lambda x)](C\cos\lambda y + D\sin\lambda y), \tag{21.15}$$

with different constants A and B.

If $\lambda^2 < 0$ then the roles of x and y interchange. The particular combination of sinusoidal and hyperbolic functions and the values of λ allowed will be determined by the geometrical properties of any specific problem, together with any prescribed or necessary boundary conditions. ◄

LAPLACE'S EQUATION; STEADY-STATE TEMPERATURE IN A RECTANGULAR PLATE

We want to solve the following problem: A long rectangular metal plate has its two long sides and the far end at 0° and the base at 100° (Figure 2.1). The width of the plate is 10 cm. Find the steady-state temperature distribution inside the plate. (This problem is mathematically identical to the problem of finding the electrostatic potential in the region $0 < x < 10, y > 0$, if the given temperatures are replaced by potentials—see, for example, Jackson, 3rd edition, p.73)

To simplify the problem, we shall assume at first that the plate is so long compared to its width that we may make the mathematical approximation that it extends to infinity in the y direction. It is then called a semi-infinite plate. This is a good approximation if we are interested in temperatures not too near the far end.

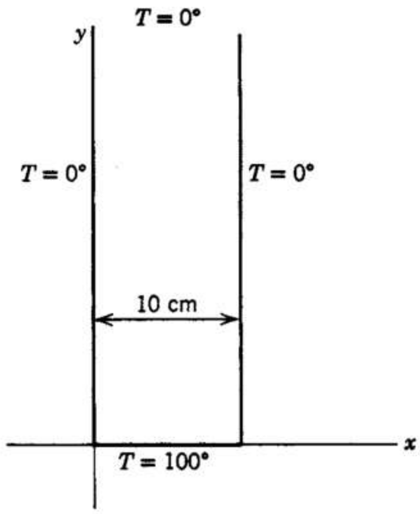

Figure 2.1

The temperature T satisfies Laplace's equation inside the plate where there are no sources of heat, that is,

$$\text{(2.1)} \qquad \nabla^2 T = 0 \qquad \text{or} \qquad \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

We have written ∇^2 in rectangular coordinates because the boundary of the plate is rectangular and we have omitted the z term because the plate is in two dimensions. To solve this equation, we are going to *try* a solution of the form

$$\text{(2.2)} \qquad T(x, y) = X(x)Y(y),$$

where, as indicated, X is a function only of x, and Y is a function only of y. Immediately you may raise the question: But how do we know that the solution is of this form? The answer is that it is not! However, as you will see, once we have solutions of the form (2.2) we can combine them to get the solution we want. [Note that a sum of solutions of (2.1) is a solution of (2.1).] Substituting (2.2) into (2.1), we have

$$\text{(2.3)} \qquad Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0.$$

(Ordinary instead of partial derivatives are now correct since X depends only on x, and y depends only on y.) Divide (2.3) by XY to get

$$\text{(2.4)} \qquad \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0.$$

The next step is really the key to the process of *separation of variables.* We are going to say that each of the terms in (2.4) is a constant because the first term is a function of x alone and the second term is a function of y alone. Why is this correct? Recall that when we say $u = \sin t$ is a *solution* of $\ddot{u} = -u$, we mean that if we substitute $u = \sin t$ into the differential equation, we get an identity ($\ddot{u} = -u$ becomes $-\sin t = -\sin t$), which is true for all values of t. Although we speak of an *equation*, when we substitute the solution into a differential equation, we have an *identity* in the independent variable. (We made use of this fact in series solutions of differential equations in Chapter 12, Sections 1 and 2.) In (2.1) to (2.4) we have two independent variables, x and y. Saying that (2.2) is a solution of (2.1) means that (2.4) is an identity in the two independent variables x and y [recall that (2.4) was obtained by substituting (2.2) into (2.1)]. In other words, if (2.2) is a solution of (2.1), then (2.4) must be true for any and all values of the two independent variables x and y. Since X is a function only of x and Y of y, the first term of (2.4) is a function only of x and the second term is a function only of y. Suppose we substitute a particular x into the first term; that term is then some numerical constant. To have (2.4) satisfied, the second term must be minus the same constant. While x remains fixed, let y vary (remember that x and y are independent). We have said that (2.4) is an identity; it is then true for our fixed x and *any* y. Thus the second term remains constant as y varies. Similarly, if we fix y and let x vary, we see that the first term of (2.4) is a constant. To say this more concisely, the equation $f(x) = g(y)$, with x and y independent variables, is an identity only if both functions are the same constant; this is the basis of the process of separation of variables. From (2.4) we then write

$$\text{(2.5)} \qquad \frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = \text{const.} = -k^2, \qquad k \geq 0, \qquad \text{or}$$
$$X'' = -k^2X \qquad \text{and} \qquad Y'' = k^2Y.$$

The constant k^2 is called the *separation constant.* The solutions of (2.5) are

$$\text{(2.6)} \qquad X = \begin{cases} \sin kx, \\ \cos kx, \end{cases} \qquad Y = \begin{cases} e^{ky}, \\ e^{-ky}, \end{cases}$$

and the solutions of (2.1) [of the form (2.2)] are

$$\text{(2.7)} \qquad T = XY = \left\{ \begin{array}{c} e^{ky} \\ e^{-ky} \end{array} \right\} \left\{ \begin{array}{c} \sin kx \\ \cos kx \end{array} \right\}.$$

None of the four solutions in (2.7) satisfies the given boundary temperatures. What we must do now is to take a combination of the solutions (2.7), with the constant k properly selected, which *will* satisfy the given boundary conditions. [Any linear combination of solutions of (2.1) is a solution of (2.1) because the differential equation (2.1) is *linear*; see Chapter 3, Section 7, and Chapter 8, Sections 1 and 6.] We first discard the solutions containing e^{ky} since we are given $T \to 0$ as $y \to \infty$. (We are assuming $k > 0$; see Problem 5.) Next we discard solutions containing $\cos kx$ since $T = 0$ when $x = 0$. This leaves us just $e^{-ky} \sin kx$, but the value of k is still to be determined. When $x = 10$, we are to have $T = 0$; this will be true if $\sin(10k) = 0$, that is, if $k = n\pi/10$ for $n = 1, 2, \cdots$. Thus for any integral n, the solution

$$T = e^{-n\pi y/10} \sin \frac{n\pi x}{10} \tag{2.8}$$

satisfies the given boundary conditions on the three $T = 0$ sides.

Finally, we must have $T = 100$ when $y = 0$; this condition is not satisfied by (2.8) for any n. But a linear combination of solutions like (2.8) is a solution of (2.1); let us try to find such a combination which does satisfy $T = 100$ when $y = 0$. In order to allow all possible n's we write an infinite series for T, namely

$$T = \sum_{n=1}^{\infty} b_n e^{-n\pi y/10} \sin \frac{n\pi x}{10}. \tag{2.9}$$

For $y = 0$, we must have $T = 100$; from (2.9) with $y = 0$ we get

$$T_{y=0} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{10} = 100. \tag{2.10}$$

Multiply both side of (2.10) by $\sin\left(\frac{m\pi x}{10}\right)$ and perform the following integration within $0 \le x \le 10$.

$$\int_0^a \sin(n\pi y/a)\, \sin(n'\pi y/a)\, dy = \begin{cases} 0, & \text{if } n' \neq n, \\ \dfrac{a}{2}, & \text{if } n' = n. \end{cases}$$

$$\int_0^a \sin(n\pi y/a)\, dy = \frac{a}{n\pi}(1 - \cos n\pi) = \begin{cases} 0, & \text{if } n \text{ is even} \\ \dfrac{2a}{n\pi}, & \text{if } n \text{ is odd.} \end{cases}$$

$$b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\, dx = \frac{2}{10}\int_0^{10} 100 \sin \frac{n\pi x}{10}\, dx = \begin{cases} \frac{400}{n\pi}, & \text{odd } n, \\ 0, & \text{even } n. \end{cases} \tag{2.11}$$

Taken from M. L. Boas

Then (2.9) becomes

$$T = \frac{400}{\pi}\left(e^{-\pi y/10}\sin\frac{\pi x}{10} + \frac{1}{3}e^{-3\pi y/10}\sin\frac{3\pi x}{10} + \cdots\right). \tag{2.12}$$

Equation (2.12) can be used for computation if $\pi y/10$ is not too small since then the series converges rapidly. (See also Problem 6.) For example, at $x = 5$ (central line of the plate) and $y = 5$, we find

$$T = \frac{400}{\pi}\left(e^{-\pi/2}\sin\frac{\pi}{2} + \frac{1}{3}e^{-3\pi/2}\sin\frac{3\pi}{2} + \cdots\right) \simeq 26.1^\circ. \tag{2.13}$$

Taken from D. J. Griffiths

Example 3.4. Two infinitely-long grounded metal plates, again at $y = 0$ and $y = a$, are connected at $x = \pm b$ by metal strips maintained at a constant potential V_0, as shown in Fig. 3.20 (a thin layer of insulation at each corner prevents them from shorting out). Find the potential inside the resulting rectangular pipe.

Solution

Once again, the configuration is independent of z. Our problem is to solve Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0,$$

subject to the boundary conditions

$$\left.\begin{array}{ll} \text{(i)} & V = 0 \text{ when } y = 0, \\ \text{(ii)} & V = 0 \text{ when } y = a, \\ \text{(iii)} & V = V_0 \text{ when } x = b, \\ \text{(iv)} & V = V_0 \text{ when } x = -b. \end{array}\right\} \tag{3.40}$$

The argument runs as before, up to Eq. 3.27:

$$V(x, y) = (Ae^{kx} + Be^{-kx})(C\sin ky + D\cos ky).$$

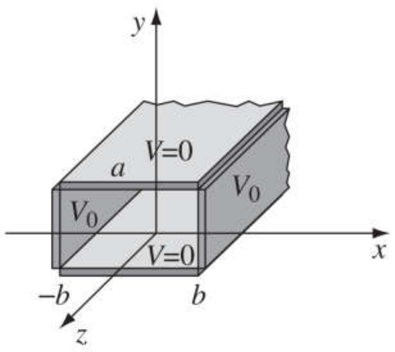

FIGURE 3.20

This time, however, we cannot set $A = 0$; the region in question does not extend to $x = \infty$, so e^{kx} is perfectly acceptable. On the other hand, the situation is *symmetric* with respect to x, so $V(-x, y) = V(x, y)$, and it follows that $A = B$. Using

$$e^{kx} + e^{-kx} = 2\cosh kx,$$

and absorbing $2A$ into C and D, we have

$$V(x, y) = \cosh kx\,(C \sin ky + D \cos ky).$$

Boundary conditions (i) and (ii) require, as before, that $D = 0$ and $k = n\pi/a$, so

$$V(x, y) = C \cosh(n\pi x/a)\, \sin(n\pi y/a). \tag{3.41}$$

Because $V(x, y)$ is even in x, it will automatically meet condition (iv) if it fits (iii). It remains, therefore, to construct the general linear combination,

$$V(x, y) = \sum_{n=1}^{\infty} C_n \cosh(n\pi x/a)\, \sin(n\pi y/a),$$

and pick the coefficients C_n in such a way as to satisfy condition (iii):

$$V(b, y) = \sum_{n=1}^{\infty} C_n \cosh(n\pi b/a)\, \sin(n\pi y/a) = V_0.$$

To obtain C_n we can use same trick as applied before. Multiply both sides by appropriate sin() function an then integrate.

$$C_n \cosh(n\pi b/a) = \begin{cases} 0, & \text{if } n \text{ is even} \\ \dfrac{4V_0}{n\pi}, & \text{if } n \text{ is odd} \end{cases}$$

Conclusion: The potential in this case is given by

$$V(x, y) = \frac{4V_0}{\pi} \sum_{n=1,3,5\ldots} \frac{1}{n} \frac{\cosh(n\pi x/a)}{\cosh(n\pi b/a)} \sin(n\pi y/a). \tag{3.42}$$

Example 3.5. An infinitely long rectangular metal pipe (sides a and b) is grounded, but one end, at $x = 0$, is maintained at a specified potential $V_0(y, z)$, as indicated in Fig. 3.22. Find the potential inside the pipe.

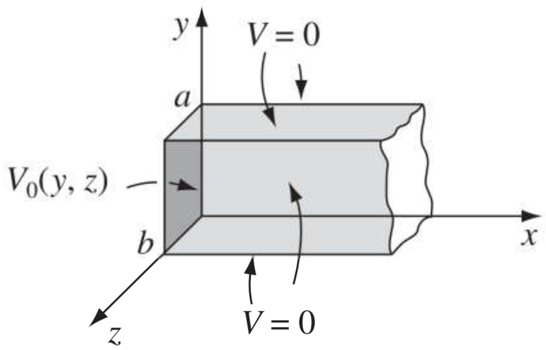

FIGURE 3.22

Solution

This is a genuinely three-dimensional problem,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0, \tag{3.43}$$

subject to the boundary conditions

$$\left.\begin{array}{ll} \text{(i)} & V = 0 \text{ when } y = 0, \\ \text{(ii)} & V = 0 \text{ when } y = a, \\ \text{(iii)} & V = 0 \text{ when } z = 0, \\ \text{(iv)} & V = 0 \text{ when } z = b, \\ \text{(v)} & V \to 0 \text{ as } x \to \infty, \\ \text{(vi)} & V = V_0(y, z) \text{ when } x = 0. \end{array}\right\} \tag{3.44}$$

As always, we look for solutions that are products:

$$V(x, y, z) = X(x)Y(y)Z(z). \tag{3.45}$$

Putting this into Eq. 3.43, and dividing by V, we find

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0.$$

It follows that

$$\frac{1}{X}\frac{d^2X}{dx^2} = C_1, \ \frac{1}{Y}\frac{d^2Y}{dy^2} = C_2, \ \frac{1}{Z}\frac{d^2Z}{dz^2} = C_3, \quad \text{with } C_1 + C_2 + C_3 = 0.$$

Our previous experience (Ex. 3.3) suggests that C_1 must be positive, C_2 and C_3 negative. Setting $C_2 = -k^2$ and $C_3 = -l^2$, we have $C_1 = k^2 + l^2$, and hence

$$\frac{d^2X}{dx^2} = (k^2 + l^2)X, \quad \frac{d^2Y}{dy^2} = -k^2 Y, \quad \frac{d^2Z}{dz^2} = -l^2 Z. \tag{3.46}$$

Once again, separation of variables has turned a *partial* differential equation into *ordinary* differential equations. The solutions are

$$\begin{aligned} X(x) &= Ae^{\sqrt{k^2+l^2}\,x} + Be^{-\sqrt{k^2+l^2}\,x}, \\ Y(y) &= C\sin ky + D\cos ky, \\ Z(z) &= E\sin lz + F\cos lz. \end{aligned}$$

Boundary condition (v) implies $A = 0$, (i) gives $D = 0$, and (iii) yields $F = 0$, whereas (ii) and (iv) require that $k = n\pi/a$ and $l = m\pi/b$, where n and m are positive integers. Combining the remaining constants, we are left with

$$V(x, y, z) = Ce^{-\pi\sqrt{(n/a)^2+(m/b)^2}\,x} \sin(n\pi y/a)\ \sin(m\pi z/b). \tag{3.47}$$

This solution meets all the boundary conditions except (vi). It contains *two* unspecified integers (n and m), and the most general linear combination is a *double* sum:

$$V(x, y, z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m} e^{-\pi\sqrt{(n/a)^2+(m/b)^2}\,x} \sin(n\pi y/a)\ \sin(m\pi z/b). \tag{3.48}$$

We hope to fit the remaining boundary condition,

$$V(0, y, z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m} \sin(n\pi y/a)\ \sin(m\pi z/b) = V_0(y, z), \tag{3.49}$$

by appropriate choice of the coefficients $C_{n,m}$. To determine these constants, we multiply by $\sin(n'\pi y/a)\ \sin(m'\pi z/b)$, where n' and m' are arbitrary positive integers, and integrate:

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m} \int_0^a \sin(n\pi y/a)\ \sin(n'\pi y/a)\, dy \int_0^b \sin(m\pi z/b)\ \sin(m'\pi z/b)\, dz \\ = \int_0^a \int_0^b V_0(y, z) \sin(n'\pi y/a)\ \sin(m'\pi z/b)\, dy\, dz.\end{aligned}$$

Quoting Eq. 3.33, the left side is $(ab/4)C_{n',m'}$, so

$$C_{n,m} = \frac{4}{ab}\int_0^a \int_0^b V_0(y, z) \sin(n\pi y/a)\ \sin(m\pi z/b)\, dy\, dz. \tag{3.50}$$

Equation 3.48, with the coefficients given by Eq. 3.50, is the solution to our problem.

For instance, if the end of the tube is a conductor at *constant* potential V_0,

$$\begin{aligned} C_{n,m} &= \frac{4V_0}{ab}\int_0^a \sin(n\pi y/a)\, dy \int_0^b \sin(m\pi z/b)\, dz \\ &= \begin{cases} 0, & \text{if } n \text{ or } m \text{ is even,} \\ \dfrac{16V_0}{\pi^2 nm}, & \text{if } n \text{ and } m \text{ are odd.} \end{cases} \end{aligned} \tag{3.51}$$

In this case

$$V(x, y, z) = \frac{16V_0}{\pi^2}\sum_{n,m=1,3,5\ldots}^{\infty} \frac{1}{nm} e^{-\pi\sqrt{(n/a)^2+(m/b)^2}\,x} \sin(n\pi y/a)\ \sin(m\pi z/b). \tag{3.52}$$

21.3 Separation of variables in polar coordinates

So far we have considered the solution of PDEs only in Cartesian coordinates, but many systems in two and three dimensions are more naturally expressed in some form of polar coordinates, in which full advantage can be taken of any inherent symmetries. For example, the potential associated with an isolated point charge has a very simple expression, $q/(4\pi\epsilon_0 r)$, when polar coordinates are used, but involves all three coordinates and square roots when Cartesians are employed. For these reasons we now turn to the separation of variables in plane polar, cylindrical polar and spherical polar coordinates.

Most of the PDEs we have considered so far have involved the operator ∇^2, e.g. the wave equation, the diffusion equation, Schrödinger's equation and Poisson's equation (and of course Laplace's equation). It is therefore appropriate that we recall the expressions for ∇^2 when expressed in polar coordinate systems. From chapter 10, in plane polars, cylindrical polars and spherical polars, respectively, we have

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}, \tag{21.23}$$

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}, \tag{21.24}$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}. \tag{21.25}$$

Of course the first of these may be obtained from the second by taking z to be identically zero.

21.3.1 Laplace's equation in polar coordinates

The simplest of the equations containing ∇^2 is Laplace's equation,

$$\nabla^2 u(\mathbf{r}) = 0. \tag{21.26}$$

Since it contains most of the essential features of the other more complicated equations, we will consider its solution first.

Laplace's equation in plane polars

Suppose that we need to find a solution of (21.26) that has a prescribed behaviour on the circle $\rho = a$ (e.g. if we are finding the shape taken up by a circular drumskin when its rim is slightly deformed from being planar). Then we may seek solutions of (21.26) that are separable in ρ and ϕ (measured from some arbitrary radius as $\phi = 0$) and hope to accommodate the boundary condition by examining the solution for $\rho = a$.

Thus, writing $u(\rho, \phi) = P(\rho)\Phi(\phi)$ and using the expression (21.23), Laplace's equation (21.26) becomes

$$\frac{\Phi}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial P}{\partial \rho}\right) + \frac{P}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} = 0.$$

Now, employing the same device as previously, that of dividing through by $u = P\Phi$ and multiplying through by ρ^2, results in the separated equation

$$\frac{\rho}{P}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial P}{\partial \rho}\right) + \frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \phi^2} = 0.$$

Following our earlier argument, since the first term on the RHS is a function of ρ only, whilst the second term depends only on ϕ, we obtain the two *ordinary* equations

$$\frac{\rho}{P}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right) = n^2, \tag{21.27}$$

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -n^2, \tag{21.28}$$

where we have taken the separation constant to have the form n^2 for later convenience; for the present, n is a general (complex) number.

Let us first consider the case in which $n \neq 0$. The second equation, (21.28), then has the general solution

$$\Phi(\phi) = A\exp(in\phi) + B\exp(-in\phi). \tag{21.29}$$

Equation (21.27), on the other hand, is the homogeneous equation

$$\rho^2 P'' + \rho P' - n^2 P = 0,$$

which must be solved either by trying a power solution in ρ or by making the substitution $\rho = \exp t$ as described in subsection 15.2.1 and so reducing it to an equation with constant coefficients. Carrying out this procedure we find

$$P(\rho) = C\rho^n + D\rho^{-n}. \tag{21.30}$$

Returning to the solution (21.29) of the azimuthal equation (21.28), we can see that if Φ, and hence u, is to be single-valued and so not change when ϕ increases by 2π then n must be an integer. Mathematically, other values of n are permissible, but for the description of real physical situations it is clear that this limitation must be imposed. Having thus restricted the possible values of n in one part of the solution, the same limitations must be carried over into the radial part, (21.30). Thus we may write a particular solution of the two-dimensional Laplace equation as

$$u(\rho, \phi) = (A \cos n\phi + B \sin n\phi)(C\rho^n + D\rho^{-n}),$$

where A, B, C, D are arbitrary constants and n is any integer.

We have not yet, however, considered the solution when $n = 0$. In this case, the solutions of the separated ordinary equations (21.28) and (21.27), respectively, are easily shown to be

$$\Phi(\phi) = A\phi + B,$$
$$P(\rho) = C \ln \rho + D.$$

But, in order that $u = P\Phi$ is single-valued, we require $A = 0$, and so the solution for $n = 0$ is simply (absorbing B into C and D)

$$u(\rho, \phi) = C \ln \rho + D.$$

Superposing the solutions for the different allowed values of n, we can write the general solution to Laplace's equation in plane polars as

$$u(\rho, \phi) = (C_0 \ln \rho + D_0) + \sum_{n=1}^{\infty} (A_n \cos n\phi + B_n \sin n\phi)(C_n\rho^n + D_n\rho^{-n}), \tag{21.31}$$

where n can take only integer values. Negative values of n have been omitted from the sum since they are already included in the terms obtained for positive n. We note that, since $\ln \rho$ is singular at $\rho = 0$, whenever we solve Laplace's equation in a region containing the origin, C_0 must be identically zero.

Laplace's equation in cylindrical polars

Passing to three dimensions, we now consider the solution of Laplace's equation in cylindrical polar coordinates,

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\phi^2}+\frac{\partial^2 u}{\partial z^2}=0. \tag{21.32}$$

We note here that, even when considering a cylindrical physical system, if there is no dependence of the physical variables on z (i.e. along the length of the cylinder) then the problem may be treated using two-dimensional plane polars, as discussed above.

For the more general case, however, we proceed as previously by trying a solution of the form

$$u(\rho,\phi,z)=P(\rho)\Phi(\phi)Z(z),$$

which, on substitution into (21.32) and division through by $u = P\Phi Z$, gives

$$\frac{1}{P\rho}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right)+\frac{1}{\Phi\rho^2}\frac{d^2\Phi}{d\phi^2}+\frac{1}{Z}\frac{d^2 Z}{dz^2}=0.$$

The last term depends only on z, and the first and second (taken together) depend only on ρ and ϕ. Taking the separation constant to be k^2, we find

$$\frac{1}{Z}\frac{d^2 Z}{dz^2}=k^2,$$

$$\frac{1}{P\rho}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right)+\frac{1}{\Phi\rho^2}\frac{d^2\Phi}{d\phi^2}+k^2=0.$$

The first of these equations has the straightforward solution

$$Z(z)=E\exp(-kz)+F\exp kz.$$

Multiplying the second equation through by ρ^2, we obtain

$$\frac{\rho}{P}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right)+\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}+k^2\rho^2=0,$$

in which the second term depends only on Φ and the other terms depend only on ρ. Taking the second separation constant to be m^2, we find

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=-m^2, \tag{21.33}$$

$$\rho\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right)+(k^2\rho^2-m^2)P=0. \tag{21.34}$$

The equation in the azimuthal angle ϕ has the very familiar solution

$$\Phi(\phi) = C\cos m\phi + D\sin m\phi.$$

As in the two-dimensional case, single-valuedness of u requires that m is an integer. However, in the particular case $m = 0$ the solution is

$$\Phi(\phi) = C\phi + D.$$

This form is appropriate to a solution with axial symmetry ($C = 0$) or one that is multivalued, but manageably so, such as the magnetic scalar potential associated with a current I (in which case $C = I/(2\pi)$ and D is arbitrary).

Finally, the ρ-equation (21.34) may be transformed into Bessel's equation of order m by writing $\mu = k\rho$. This has the solution

$$P(\rho) = AJ_m(k\rho) + BY_m(k\rho).$$

The properties of these functions were investigated in chapter 16 and will not be pursued here. We merely note that $Y_m(k\rho)$ is singular at $\rho = 0$, and so, when seeking solutions to Laplace's equation in cylindrical coordinates within some region containing the $\rho = 0$ axis, we require $B = 0$.

The complete separated-variable solution in cylindrical polars of Laplace's equation $\nabla^2 u = 0$ is thus given by

$$u(\rho, \phi, z) = [AJ_m(k\rho) + BY_m(k\rho)][C\cos m\phi + D\sin m\phi][E\exp(-kz) + F\exp kz]. \tag{21.35}$$

Of course we may use the principle of superposition to build up more general solutions by adding together solutions of the form (21.35) for all allowed values of the separation constants k and m.

Laplace's equation in spherical polars

We now come to an equation that is very widely applicable in physical science, namely $\nabla^2 u = 0$ in spherical polar coordinates:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = 0. \qquad (21.38)$$

Our method of procedure will be as before; we try a solution of the form

$$u(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi).$$

Substituting this in (21.38), dividing through by $u = R\Theta\Phi$ and multiplying by r^2, we obtain

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi\sin^2\theta}\frac{d^2\Phi}{d\phi^2} = 0. \qquad (21.39)$$

The first term depends only on r and the second and third terms (taken together) depend only on θ and ϕ. Thus (21.39) is equivalent to the two equations

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \lambda, \qquad (21.40)$$

$$\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi\sin^2\theta}\frac{d^2\Phi}{d\phi^2} = -\lambda. \qquad (21.41)$$

Equation (21.40) is a homogeneous equation,

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - \lambda R = 0,$$

which can be reduced, by the substitution $r = \exp t$ (and writing $R(r) = S(t)$), to

$$\frac{d^2S}{dt^2} + \frac{dS}{dt} - \lambda S = 0.$$

This has the straightforward solution

$$S(t) = A\exp\lambda_1 t + B\exp\lambda_2 t,$$

and so the solution to the radial equation is

$$R(r) = Ar^{\lambda_1} + Br^{\lambda_2},$$

where $\lambda_1 + \lambda_2 = -1$ and $\lambda_1\lambda_2 = -\lambda$. We can thus take λ_1 and λ_2 as given by ℓ and $-(\ell+1)$; λ then has the form $\ell(\ell+1)$. (It should be noted that at this stage nothing has been either assumed or proved about whether ℓ is an integer.)

Hence we have obtained some information about the first factor in the separated-variable solution, which will now have the form

$$u(r, \theta, \phi) = \left[Ar^{\ell} + Br^{-(\ell+1)} \right] \Theta(\theta)\Phi(\phi), \tag{21.42}$$

where Θ and Φ must satisfy (21.41) with $\lambda = \ell(\ell + 1)$.

The next step is to take (21.41) further. Multiplying through by $\sin^2\theta$ and substituting for λ, it too takes a separated form:

$$\left[\frac{\sin\theta}{\Theta} \frac{d}{d\theta} \left(\sin\theta \frac{d\Theta}{d\theta} \right) + \ell(\ell+1)\sin^2\theta \right] + \frac{1}{\Phi} \frac{d^2\Phi}{d\phi^2} = 0. \tag{21.43}$$

Taking the separation constant as m^2, the equation in the azimuthal angle ϕ has the same solution as in cylindrical polars, namely

$$\Phi(\phi) = C\cos m\phi + D\sin m\phi.$$

As before, single-valuedness of u requires that m is an integer; for $m = 0$ we again have $\Phi(\phi) = C\phi + D$.

Having settled the form of $\Phi(\phi)$, we are left only with the equation satisfied by $\Theta(\theta)$, which is

$$\frac{\sin\theta}{\Theta} \frac{d}{d\theta} \left(\sin\theta \frac{d\Theta}{d\theta} \right) + \ell(\ell+1)\sin^2\theta = m^2. \tag{21.44}$$

A change of independent variable from θ to $\mu = \cos\theta$ will reduce this to a form for which solutions are known, and of which some study has been made in chapter 16. Putting

$$\mu = \cos\theta, \qquad \frac{d\mu}{d\theta} = -\sin\theta, \qquad \frac{d}{d\theta} = -(1-\mu^2)^{1/2}\frac{d}{d\mu},$$

the equation for $M(\mu) \equiv \Theta(\theta)$ reads

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{dM}{d\mu} \right] + \left[\ell(\ell+1) - \frac{m^2}{1-\mu^2} \right] M = 0. \tag{21.45}$$

This equation is the *associated Legendre equation*, which was mentioned in subsection 18.2 in the context of Sturm–Liouville equations.

We recall that for the case $m = 0$, (21.45) reduces to Legendre's equation, which was studied at length in chapter 16, and has the solution

$$M(\mu) = EP_{\ell}(\mu) + FQ_{\ell}(\mu). \tag{21.46}$$

We have not solved (21.45) explicitly for general m, but the solutions were given in subsection 18.2 and are the associated Legendre functions $P_\ell^m(\mu)$ and $Q_\ell^m(\mu)$, where

$$P_\ell^m(\mu) = (1-\mu^2)^{|m|/2} \frac{d^{|m|}}{d\mu^{|m|}} P_\ell(\mu), \tag{21.47}$$

and similarly for $Q_\ell^m(\mu)$. We then have

$$M(\mu) = EP_\ell^m(\mu) + FQ_\ell^m(\mu); \tag{21.48}$$

here m must be an integer, $0 \leq |m| \leq \ell$. We note that if we require solutions to Laplace's equation that are finite when $\mu = \cos\theta = \pm 1$ (i.e. on the polar axis where $\theta = 0, \pi$), then we must have $F = 0$ in (21.46) and (21.48) since $Q_\ell^m(\mu)$ diverges at $\mu = \pm 1$.

It will be remembered that one of the important conditions for obtaining finite polynomial solutions of Legendre's equation is that ℓ is an integer ≥ 0. This condition therefore applies also to the solutions (21.46) and (21.48) and is reflected back into the radial part of the general solution given in (21.42).

Now that the solutions of each of the three ordinary differential equations governing R, Θ and Φ have been obtained, we may assemble a complete separated-variable solution of Laplace's equation in spherical polars. It is

$$u(r, \theta, \phi) = (Ar^\ell + Br^{-(\ell+1)})(C\cos m\phi + D\sin m\phi)[EP_\ell^m(\cos\theta) + FQ_\ell^m(\cos\theta)], \tag{21.49}$$

where the three bracketted factors are connected only through the *integer* parameters ℓ and m, $0 \leq |m| \leq \ell$. As before, a general solution may be obtained by superposing solutions of this form for the allowed values of the separation constants ℓ and m. As mentioned above, if the solution is required to be finite on the polar axis then $F = 0$ for all ℓ and m.

3.3.2 ■ Spherical Coordinates

In the examples considered so far, Cartesian coordinates were clearly appropriate, since the boundaries were planes. For round objects, spherical coordinates are more natural. In the spherical system, Laplace's equation reads:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}=0. \tag{3.53}$$

I shall assume the problem has **azimuthal symmetry**, so that V *is independent of* ϕ;[12] in that case, Eq. 3.53 reduces to

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)=0. \tag{3.54}$$

As before, we look for solutions that are products:

$$V(r,\theta)=R(r)\Theta(\theta). \tag{3.55}$$

Putting this into Eq. 3.54, and dividing by V,

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=0. \tag{3.56}$$

Since the first term depends only on r, and the second only on θ, it follows that each must be a constant:

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=l(l+1),\qquad \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=-l(l+1). \tag{3.57}$$

Here $l(l+1)$ is just a fancy way of writing the separation constant—you'll see in a minute why this is convenient.

As always, separation of variables has converted a *partial* differential equation (3.54) into *ordinary* differential equations (3.57). The radial equation,

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=l(l+1)R, \tag{3.58}$$

has the general solution

$$R(r)=Ar^l+\frac{B}{r^{l+1}}, \tag{3.59}$$

as you can easily check; A and B are the two arbitrary constants to be expected in the solution of a second-order differential equation. But the angular equation,

$$\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -l(l+1)\sin\theta\,\Theta, \tag{3.60}$$

is not so simple. The solutions are **Legendre polynomials** in the variable $\cos\theta$:

$$\Theta(\theta) = P_l(\cos\theta). \tag{3.61}$$

$P_l(x)$ is most conveniently defined by the **Rodrigues formula:**

$$P_l(x) \equiv \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l. \tag{3.62}$$

The first few Legendre polynomials are listed in Table 3.1.

$P_0(x)$	$=$	1
$P_1(x)$	$=$	x
$P_2(x)$	$=$	$(3x^2-1)/2$
$P_3(x)$	$=$	$(5x^3-3x)/2$
$P_4(x)$	$=$	$(35x^4-30x^2+3)/8$
$P_5(x)$	$=$	$(63x^5-70x^3+15x)/8$

TABLE 3.1 Legendre Polynomials.

Notice that $P_l(x)$ is (as the name suggests) an lth-order *polynomial* in x; it contains only *even* powers, if l is even, and *odd* powers, if l is odd. The factor in front $(1/2^l l!)$ was chosen in order that

$$P_l(1) = 1. \tag{3.63}$$

The Rodrigues formula obviously works only for nonnegative integer values of l. Moreover, it provides us with only *one* solution. But Eq. 3.60 is *second*-order, and it should possess *two* independent solutions, for *every* value of l. It turns out that these "other solutions" blow up at $\theta = 0$ and/or $\theta = \pi$, and are therefore unacceptable on physical grounds.[13] For instance, the second solution for $l = 0$ is

$$\Theta(\theta) = \ln\left(\tan\frac{\theta}{2}\right). \tag{3.64}$$

You might want to check for yourself that this satisfies Eq. 3.60.

In the case of azimuthal symmetry, then, the most general separable solution to Laplace's equation, consistent with minimal physical requirements, is

$$V(r, \theta) = \left(Ar^l + \frac{B}{r^{l+1}} \right) P_l(\cos\theta).$$

(There was no need to include an overall constant in Eq. 3.61 because it can be absorbed into A and B at this stage.) As before, separation of variables yields an infinite set of solutions, one for each l. The *general* solution is the linear combination of separable solutions:

$$\boxed{V(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta). \qquad (3.65)}$$

The following examples illustrate the power of this important result.

Example 3.8. An uncharged metal sphere of radius R is placed in an otherwise uniform electric field $\mathbf{E} = E_0\hat{\mathbf{z}}$. The field will push positive charge to the "northern" surface of the sphere, and—symmetrically—negative charge to the "southern" surface (Fig. 3.24). This induced charge, in turn, distorts the field in the neighborhood of the sphere. Find the potential in the region outside the sphere.

Solution

The sphere is an equipotential—we may as well set it to zero. Then by symmetry the entire xy plane is at potential zero. This time, however, V does *not* go to zero at large z. In fact, far from the sphere the field is $E_0\hat{\mathbf{z}}$, and hence

$$V \to -E_0 z + C.$$

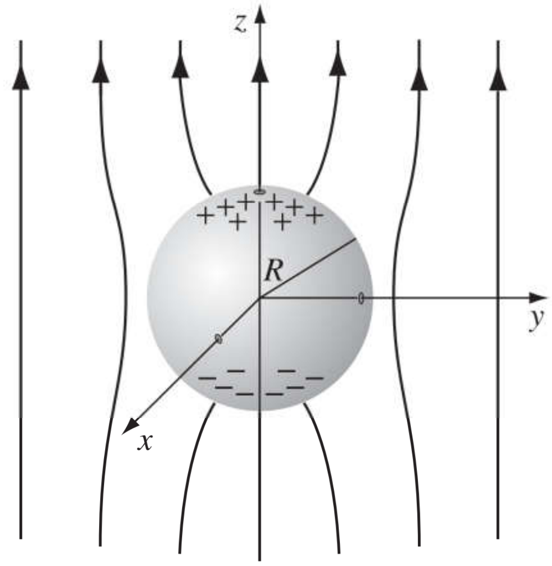

FIGURE 3.24

Since $V = 0$ in the equatorial plane, the constant C must be zero. Accordingly, the boundary conditions for this problem are

$$\left.\begin{array}{ll}\text{(i)} \quad V = 0 & \text{when } r = R, \\ \text{(ii)} \quad V \to -E_0 r \cos\theta & \text{for } r \gg R.\end{array}\right\} \tag{3.74}$$

We must fit these boundary conditions with a function of the form 3.65.

The first condition yields

$$A_l R^l + \frac{B_l}{R^{l+1}} = 0,$$

or

$$B_l = -A_l R^{2l+1}, \tag{3.75}$$

so

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l \left(r^l - \frac{R^{2l+1}}{r^{l+1}} \right) P_l(\cos\theta).$$

For $r \gg R$, the second term in parentheses is negligible, and therefore condition (ii) requires that

$$\sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) = -E_0 r \cos\theta.$$

Evidently only one term is present: $l = 1$. In fact, since $P_1(\cos\theta) = \cos\theta$, we can read off immediately

$$A_1 = -E_0, \quad \text{all other } A_l\text{'s zero.}$$

Conclusion:

$$V(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta. \tag{3.76}$$

The first term $(-E_0 r\cos\theta)$ is due to the external field; the contribution attributable to the induced charge is

$$E_0\frac{R^3}{r^2}\cos\theta.$$

If you want to know the induced charge density, it can be calculated in the usual way:

$$\sigma(\theta) = -\epsilon_0\frac{\partial V}{\partial r}\bigg|_{r=R} = \epsilon_0 E_0\left(1 + 2\frac{R^3}{r^3}\right)\cos\theta\bigg|_{r=R} = 3\epsilon_0 E_0\cos\theta. \tag{3.77}$$

As expected, it is positive in the "northern" hemisphere $(0 \le \theta \le \pi/2)$ and negative in the "southern" $(\pi/2 \le \theta \le \pi)$.
