# Oscillatory motion and related physics

## Contents

# 1 Introduction

Oscillatory motion is a very special kind of motion in which the moving body repeats its trajectory in to and fro fashion. In fact whenever a particle is disturbed slightly from its stable equilibrium point, it exhibits oscillatory motion. Therefore, it is very important to understand the mechanical equilibrium in connection with oscillatory motion.

## 1.1 Equilibrium

A particle is said to be in equilibrium at some point in space if net force acting on the particle vanishes exactly at that specified point. The point is known to be equilibrium point. In a given potential function $V(\mathbf{r})$, it is possible to find out the equilibrium points. According to definition of equilibrium, at equilibrium points,

$$\mathbf{F} = -\nabla V(\mathbf{r}) = 0$$

Let us consider one dimensional case, then at equilibrium point $x_0$,

$$F(x_0) = -\left(\frac{dV(x)}{dx}\right)_{x_0} = 0$$

Since the first derivative of $V(x)$ vanishes at equilibrium points, that means the potential function will exhibit extrema (maxima or minima) at equilibrium points. *If it shows minima then the equilibrium will be stable equilibrium and if it shows maxima the equilibrium will be unstable equilibrium.* Consider the figure

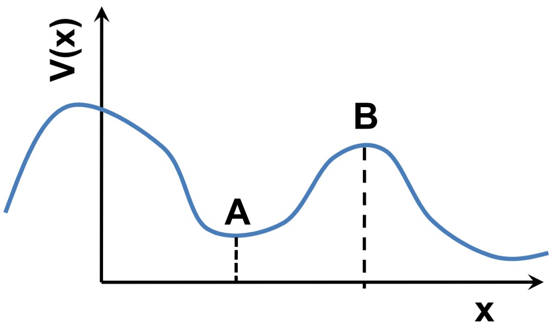


Figure 1: Potential function in one dimension and equilibrium points.

1, in which the variation of one dimentional potential function over space is shown. $F(x) = -\frac{dV(x)}{dx}$ vanishes at two points at A and B. That means both A and B are equilibrium point; but the stable equilibrium is at point A. If a body is slightly displaced from A, then it tries to minimize its potential an turns back to point A. On the other hand if a body is slightly displaced from point B, it also tries to minimize its potential so that it never goes back to point B. Therefore, when a particle is disturbed about point A it exhibits oscillatory motion.

For small displacement from equilibrium point $x_0$, the potential can be expressed in Taylor series expansion as,

$$V(x) = V(x_0) + \left(\frac{dV}{dx}\right)_{x_0}(x - x_0) + \frac{1}{2}\left(\frac{d^2V}{dx^2}\right)_{x_0}(x - x_0)^2 + \frac{1}{3!}\left(\frac{d^3V}{dx^3}\right)_{x_0}(x - x_0)^3 + \dots\dots\dots\dots\dots \quad (1)$$

But at equilibrium point,

$$F(x_0) = \left(\frac{dV}{dx}\right)_{x_0} = 0$$

The additive constant potential $V(x_0)$ does not have any role in describing the motion. Therefore, for small displacement about $x_0$ the lowest order non-vanishing term is quadratic and we can neglect the higher order terms. That means, equation (1) can be approximated as,

$$V(x) = \frac{1}{2}\left(\frac{d^2V}{dx^2}\right)_{x_0}(x - x_0)^2 \tag{2}$$

Motion of a particle in this potential is known as *simple harmonic motion*. Further we can set our origin at $x_0$ such that the potential takes the form,

$$\boxed{V(x) = \frac{1}{2}\left(\frac{d^2V}{dx^2}\right)_0 x^2 = \frac{1}{2}kx^2} \tag{3}$$

where $k$ is known as *spring constant*.

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• **Problem :** Consider one dimensional potential,

$$V(x) = 4\sigma\left(\frac{1}{x^6} - \frac{1}{x^{12}}\right)$$

Plot the variation of $V(x)$. Find the equilibrium position and check whether it is stable or unstable equilibrium. If a particle is slightly displaced from its equilibrium point then determine the spring constant associated with its motion.

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## 2 Simple harmonic motion (SHM)

Potential energy of an one dimensional simple harmonic oscillator performing oscillation about origin is given by equation (3). Hence the force acting on the particle is,

$$F(x) = -\frac{dV}{dx} = -kx$$

Therefore, the equation of motion becomes,

$$\begin{aligned} m\frac{d^2x}{dt^2} &= -kx \\ \Rightarrow \quad \frac{d^2x}{dt^2} + \omega_0^2 x &= 0 \end{aligned} \tag{4}$$

where $\omega_0 = \sqrt{k/m}$ is known as *natural frequency* of the harmonic motion. The solution of this second order differential equation can be expressed as,

$$\begin{aligned} x(t) &= A\sin(\omega_0 t + \phi) \\ &\text{or} \\ &= A\cos(\omega_0 t + \phi) \end{aligned} \tag{5}$$

Two arbitrary constants $A$ and $\phi$ associated with 2nd order differential equation are known as *amplitude* and *phase angle* respectively. They are to be determined from initial (or boundary) conditions. For simplicity we can set the initial phase angle $\phi = 0$ without loss of any generality.

### 2.1 Energy of SHM

Let consider the SHM described as,

$$x(t) = A\sin(\omega_0 t)$$

Therefore, instantaneous velocity of the oscillator is given by,

$$v(t) = \frac{dx}{dt} = \omega_0 A\cos(\omega_0 t)$$

Potential energy,

$$E_P = \frac{1}{2}kx^2 = \frac{1}{2}m\omega_0^2 x^2 = \frac{1}{2}m\omega_0^2 A^2\sin^2(\omega_0 t)$$

Kinetic energy,

$$E_K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega_0^2 A^2\cos^2(\omega_0 t)$$

Total energy,

$$E = E_P + E_K = \frac{1}{2}m\omega_0^2 A^2\sin^2(\omega_0 t) + \frac{1}{2}m\omega_0^2 A^2\cos^2(\omega_0 t) = \frac{1}{2}m\omega_0^2 A^2$$

Although $E_P$ and $E_K$ are time dependent but total energy is constant with respect to time. Actually, during the motion, $E_K$ and $E(P)$ are changing such that their sum remains constant.

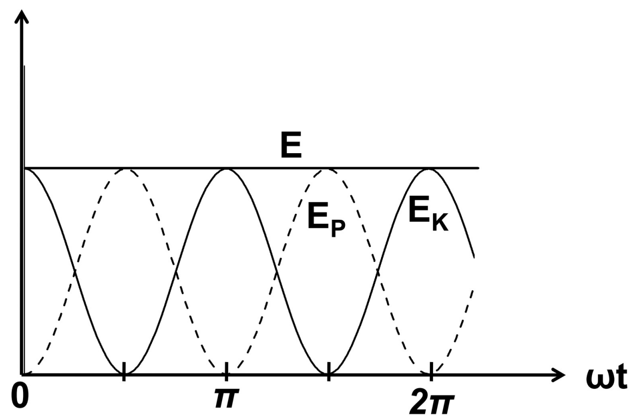


Figure 2: Variation of kinetic energy, potential energy and total energy of a particle executing SHM. Total energy s constant.

## 3 Damped harmonic motion

An oscillating particle in general feels resistive force during its oscillatory motion due to viscous drag of the medium in which it is oscillating. The resistive force can be approximated to be proportional to its instantaneous velocity and acting against its motion. In this situation the equation of motion of oscillating particle can be expressed as,

$$m\frac{d^2x}{dt^2} = -\lambda\frac{dx}{dt} - kx$$

where, $-\lambda\frac{dx}{dt}$ is the resistive force proportional to instantaneous velocity $\frac{dx}{dt}$ with proportionality constant $\lambda$. $-kx$ is the restoring force responsible for SHM. The equation of motion can be rewritten as,

$$\boxed{\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = 0} \tag{6}$$

where, $\gamma = \lambda/m$ and $\omega_0^2 = k/m$

Trial solution of this equation is,

$$x(t) \sim e^{\beta t}$$

substituting this trial solution in equation (6) we get,

$$\begin{aligned} & \beta^2 + \gamma\beta + \omega_0^2 = 0 \\ \Rightarrow \quad & \beta = \frac{1}{2}[-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}] \\ \Rightarrow \quad & \beta = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \qquad ; \quad \text{Let, } \alpha = \gamma/2 \end{aligned} \tag{7}$$

Depending upon the value of the factor $\sqrt{\alpha^2 - \omega_0^2}$ we can get various situations of motion.

• **Case-I :** Let $(\alpha^2 - \omega_0^2) > 0$; In this case the motion is non oscillatory known as *over damped motion.* The solution is given by,

$$x(t) = e^{-\alpha t}\left[A \exp\left(t\sqrt{\alpha^2 - \omega_0^2}\right) + B \exp\left(-t\sqrt{\alpha^2 - \omega_0^2}\right)\right]$$

• **Case-II :** Let $(\alpha^2 - \omega_0^2) = 0$; In this case the motion is non oscillatory known as *critically damped motion.* The solution is given by,

$$x(t) = e^{-\alpha t}(At + B)$$

• **Case-III :** Let $(\alpha^2 - \omega_0^2) < 0$; In this case the motion is oscillatory and **we are interested to explore this situation.**

We have

$$\beta = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

with $(\alpha^2 - \omega_0^2) < 0$. Therefore, $\beta$ becomes a complex number,

$$\boxed{\beta = -\alpha \pm i\omega}$$

where we define,

$$\boxed{\omega = \sqrt{\omega_0^2 - \alpha^2}} \tag{8}$$

Since the form of solution is,

$$x(t) \sim e^{\beta t}$$

We have the solution for this case,

$$\begin{aligned} x(t) &= e^{-\alpha t}(ae^{i\omega t} + be^{-i\omega t}) \\ &= e^{-\alpha t}[a(\cos\omega t + i\sin\omega t) + b(\cos\omega t - i\sin\omega t)] \\ &= e^{-\alpha t}[(a+b)\cos\omega t + i(a-b)\sin\omega t)] \\ &= Ae^{-\alpha t}\cos(\omega t + \phi) \end{aligned} \tag{9}$$

where, $A\cos\phi = (a+b)$ and $A\cos\phi = -i(a-b)$. In fact, we can choose the initial condition such that $\phi = 0$. Therefore, the general solution of *damped harmonic oscillation* is given by,

$$\boxed{x(t) = Ae^{-\alpha t}\cos(\omega t)} \tag{10}$$

To be noted that the particle oscillates with frequency $\omega = \sqrt{\omega_0^2 - \alpha^2}$ instead of its natural frequency $\omega_0$. However, it can be shown that in weak damping i.e., $\alpha << \omega_0$, the particle oscillates almost with its natural frequency $\omega \approx \omega_0$. For damped oscillation, the amplitude falls off exponentially with time due to the factor $e^{-\alpha t}$.

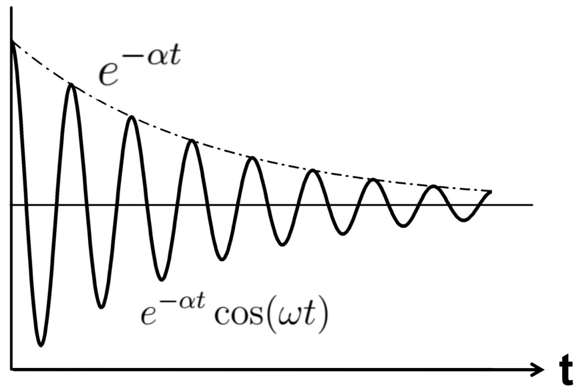


Figure 3: Time dependent displacement of damped oscillator.

### 3.1 Energy of damped oscillator

Potential energy,

$$E_P = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 x^2 = \frac{1}{2}m\omega^2 A^2 e^{-2\alpha t}\cos^2(\omega t)$$

To determine the kinetic energy, we have to compute velocity of the oscillator. From expression (10),

$$v(t) = \frac{dx}{dt} = Ae^{-\alpha t}[-\alpha\cos\omega t - \omega\sin\omega t]$$

Mostly we encounter weak dissipation i.e; $\alpha << \omega$, therefore,

$$\frac{dx}{dt} \approx -\omega A e^{-\alpha t}\sin\omega t = \omega A e^{-\alpha t}\cos(\omega t + \pi/2)$$

It is to be noted for small damping, velocity leads the displacement by $\pi/2$ phase angle.

Now the kinetic energy,

$$E_K = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 = \frac{1}{2}m\omega^2 A^2 e^{-2\alpha t}\sin^2(\omega t)$$

Total energy,

$$E = (E_P + E_K) = \frac{1}{2}m\omega^2 A^2 e^{-2\alpha t}\cos^2(\omega t) + \frac{1}{2}m\omega^2 A^2 e^{-2\alpha t}\sin^2(\omega t) \quad = \quad \frac{1}{2}m\omega^2 A^2 e^{-2\alpha t}$$

Therefore, total energy of damped harmonic oscillator decreases exponentially with time.

Here it is easy to show from the final expressions of $E_K$, $E_P$ and $E$,

$$\langle E_K\rangle = \langle E_P\rangle = E/2$$

### 3.2 Logarithmic decrement

Amplitude of damped oscillator decreases with time as $Ae^{-\alpha t}$. Suppose, $d_0$, $d_1$, $d_2$, $d_3$, ....... are the successive maximum displacements about mean position in both directions. That means, for a positive integer variable $n$, $d_{2n}$ express maximum displacements in one side and $d_{2n+1}$ express maximum displacements in another side. Therefore, if the time period of oscillation is $t$ then,

$$d_0 = A \quad , \quad d_1 = A\exp(-\alpha T/2)$$

$$d_2 = A\exp(-\alpha T) \quad , \quad d_3 = A\exp(-3\alpha T/2)$$

$$\dots\dots\dots\dots\dots\dots$$

$$d_{2n} = A\exp(-n\alpha T) \quad , \quad d_{2n+1} = A\exp[-(2n+1)\alpha T/2]$$

Therefore,

$$\frac{d_{2n}}{d_{2n+1}} = \exp(\alpha T/2) \qquad ; \quad n = 0, 1, 2, 3, \dots\dots$$

The *logarithmic decrement* $\Lambda$ is defined as,

$$\Lambda = \ln\left(\frac{d_{2n}}{d_{2n+1}}\right)$$

Therefore, using the above expressions we have,

$$\Lambda = \alpha T/2$$

### 3.3 Equation of motion from energy conservation

Damped oscillator continuously dissipates it total energy due to resistive force. The rate of change of total energy must be equal to loss rate of energy due to damping.

$$\text{Resistive force} = m\gamma\frac{dx}{dt}$$

Therefore, work done due to infinitesimal displacement $dx$ under this force,

$$dW = m\gamma\frac{dx}{dt}\ dx$$

Therefore, energy loss rate due to damping,

$$\frac{dW}{dt} = m\gamma\left(\frac{dx}{dt}\right)^2$$

According to energy conservation,

$$\begin{aligned}
&-\frac{dE}{dt} = \frac{dW}{dt} \\
\Rightarrow\quad &\frac{d}{dt}(E_P + E_K) = -m\gamma\left(\frac{dx}{dt}\right)^2 \\
\Rightarrow\quad &\frac{d}{dt}\left(\frac{1}{2}kx^2 + \frac{1}{2}m\dot{x}^2\right) = -m\gamma\dot{x}^2 \\
\Rightarrow\quad &kx\dot{x} + m\dot{x}\ddot{x} + m\gamma\dot{x}^2 = 0 \\
\Rightarrow\quad &\ddot{x} + \gamma\dot{x} + (k/m)x = 0 \\
\Rightarrow\quad &\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0
\end{aligned} \tag{11}$$

Hence, we have obtained the equation of motion identical to equation (6)

# 4 Forced harmonic oscillation

In this case an oscillator is driven externally by a periodic (sinusoidal) force can be expressed as $F_0 \cos(\omega t)$, where $F_0$ is the amplitude of the force. The equation of motion is given as,

$$m\ddot{x} + \lambda\dot{x} + kx = F_0 \cos(\omega t)$$

or,

$$\boxed{\ddot{x} + \gamma\dot{x} + \omega_0^2 x = f_0 \cos(\omega t)} \tag{12}$$

where, $\gamma = \lambda/m, \quad \omega_0 = \sqrt{k/m}$ and $\quad f_0 = F_0/m$

The equation of motion (12) can be solved easily by introducing the complex quantity $e^{i\omega t}$ which is periodic function in complex plane and moreover it can be decomposed as,

$$\exp(i\omega t) = \cos(\omega t) + i\sin(\omega t) \qquad ; \quad i = \sqrt{-1}$$

That means sinusoidal oscillation can be captured through the function $\exp(i\omega t)$. Hence, the equation of motion becomes,

$$\boxed{\ddot{x} + \gamma\dot{x} + \omega_0^2 x = f_0 \exp(i\omega t)} \tag{13}$$

It is obvious that due to external force, the oscillator will exhibit a harmonic oscillation with frequency equals to the frequency of applied force i.e; $\omega$ instead of its natural frequency $\omega_0$.

Let us consider the solution of equation (13) as,

$$x(t) = A\exp(i\omega t) \tag{14}$$

where $A$ is the amplitude of oscillation which is in this case a complex quantity. Putting this expression of $x(t)$ in equation (13),

$$\begin{aligned} & A[-\omega^2 + i\gamma\omega + \omega_0^2] = f_0 \\ \Rightarrow \quad & A(\omega) = \frac{f_0}{(\omega_0^2 - \omega^2) + i\gamma\omega} = \frac{f_0}{Z} \end{aligned} \tag{15}$$

Where,

$$Z = (\omega_0^2 - \omega^2) + i\gamma\omega = |Z|\exp(i\phi)$$

where,

$$|Z| = \sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} \qquad \text{and} \qquad \tan\phi = \frac{\gamma\omega}{\omega_0^2 - \omega^2} \tag{16}$$

Therefore,

$$A(\omega) = \frac{f_0}{Z} = \frac{f_0}{|Z|\exp(i\phi)} = \frac{f_0}{|Z|}\exp(-i\phi)$$

Now from expression (14) we get,

$$x(t) = A(\omega)\exp(i\omega t) = \frac{f_0}{|Z|}\exp[i(\omega t - \phi)]$$

If we consider the force as $F(t) = F_0\cos(\omega t)$ then the solution becomes,

$$\boxed{x(t) = \frac{f_0}{|Z|}\cos(\omega t - \phi)} \tag{17}$$

where $|Z|$ and $\phi$ are given in expressions (16).

### 4.1 Energy of forced oscillator

Potential energy,

$$E_P = \frac{1}{2}m\omega^2 x^2 = \frac{m\omega^2 f_0^2}{2|Z|^2}\cos^2(\omega t - \phi) \qquad ; \quad \text{using (17)}$$

Kinetic energy,

$$E_K = \frac{1}{2}m\dot{x}^2 = \frac{m\omega^2 f_0^2}{2|Z|^2}\sin^2(\omega t - \phi) \qquad ; \quad \text{using (17)}$$

Therefore, total energy

$$E = (E_P + E_K) = \frac{m\omega^2 f_0^2}{2|Z|^2}$$

Both kinetic energy and potential energy are oscillatory with time but the total energy energy is constant for forced harmonic oscillator. In forced harmonic oscillator an equilibrium situation is attained between energy delivered by external force and energy loss due to damping. The rate of energy input by driving force exactly balanced by the rate of energy loss due to dissipation; so that total energy remains constant.

Instantaneous power delivered by external force,

$$p_{in}(t) = F(t)\dot{x}(t) = \dot{x}mf_0\cos(\omega t)$$

Evaluating $\dot{x}$ using expression (17) ,

$$p_{in}(t) = -\frac{m\omega f_0^2}{|Z|}\cos(\omega t)\sin(\omega t - \phi) = -\frac{m\omega f_0^2}{|Z|}[\sin(\omega t)\cos(\omega t)\cos\phi - \cos^2(\omega t)\sin\phi]$$

Therefore, average power delivered by driving force,

$$\langle p_{in}(t)\rangle = \frac{m\omega f_0^2}{2|Z|}\sin\phi$$

where we have used the fact,

$$\langle\sin(\omega t)\cos(\omega t)\rangle = 0 \qquad \text{and} \qquad \langle\cos^2(\omega t)\rangle = 1/2$$

Now,

$$\sin\phi = \frac{\tan\phi}{\sqrt{1+\tan^2\phi}} = \frac{\gamma\omega}{\sqrt{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}} = \frac{\gamma\omega}{|Z|} \quad ; \quad \text{using equation (16)}$$

Therefore,

$$\boxed{\langle p_{in}(t)\rangle = \frac{m\omega f_0^2}{|Z|}\sin\phi = \frac{m\gamma\omega^2 f_0^2}{2|Z|^2}} \tag{18}$$

This is the expression for average power delivered to the oscillator by the driving force.

Now we evaluate the average power dissipation by damping force;

$$p_{out}(t) = \text{damping force} \times \text{velocity} = m\gamma\dot{x}^2 = \frac{m\gamma\omega^2 f_0^2}{|Z|^2}\sin^2(\omega t - \phi) \quad ; \quad \text{Using (17)}$$

Therefore,

$$\boxed{\langle p_{out}(t)\rangle = \frac{m\gamma\omega^2 f_0^2}{2|Z|^2}} \qquad ; \quad \text{Since,} \quad \langle\sin^2(\omega t - \phi)\rangle = 1/2$$

That means average power delivered by driving force is identical to average power dissipation by damping force such that total energy remains constant with time.

## 4.2 Resonance phenomena

### 4.2.1 Amplitude resonance

From expression (17), the amplitude of oscillation is observed to be frequency dependent which is given by,

$$|A(\omega)| = \frac{f_0}{|Z|} = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}$$

The amplitude becomes maximum at some driving frequency $\omega_r$ and this phenomena is known as *amplitude resonance*. Therefore at amplitude resonance frequency $\omega_r$ the denominator should be minimum; that means,

$$\begin{aligned}
&\frac{d}{d\omega}[(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2)]_{\omega_r} = 0 \\
\Rightarrow \quad & -4\omega_r(\omega_0^2 - \omega_r^2) + 2\gamma^2\omega_r^2 = 0 \\
\Rightarrow \quad & \omega_r = (\omega_0^2 - \gamma^2/2)^{1/2} = \omega_0\left(1 - \gamma^2/2\omega_0^2\right)^{1/2}
\end{aligned} \tag{19}$$

Therefore, amplitude resonance occurs at slightly less frequency than that of natural frequency $\omega_0$. However in the weak damping case ($\gamma << \omega_0$), where, amplitude resonance frequency $\omega_r \approx \omega_0$.

### 4.2.2 Velocity resonance

Using the expression for displacement given in equation (17),

$$v(t) = \dot{x}(t) = -\frac{\omega f_0}{|Z|}\sin(\omega t - \phi)$$

Therefore, velocity amplitude of oscillator is,

$$V(\omega) = \frac{\omega f_0}{|Z|} = \frac{\omega f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}} = \frac{f_0}{\sqrt{(\omega_0^2/\omega - \omega)^2 + \gamma^2}}$$

The velocity resonance occurs at some particular frequency $\omega'$ at which $V(\omega)$ becomes maximum; that means the denominator factor in the expression of $V(\omega)$ becomes minimum at $\omega = \omega'$. The criteria of minimum value of denominator is,

$$\omega_0^2/\omega' - \omega' = 0 \qquad \Rightarrow \quad \omega' = \omega_0$$

Therefore, when the frequency of driving force exactly matches with the natural frequency ($\omega_0$) of the oscillator then velocity resonance occurs.

### 4.2.3 Power resonance

As we have already seen that from energy conservation the average power delivered to the oscillator by driving force exactly matches to the average power dissipation due to damping force that means $\langle p_{out}(t)\rangle = \langle p_{in}(t)\rangle$. Input power as well as consumed power obtained above as,

$$\langle p_{out}(t)\rangle = \langle p_{in}(t)\rangle = \frac{m\gamma\omega^2 f_0^2}{2|Z|^2} = \frac{m\gamma\omega^2 f_0^2}{2[(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2]} = \frac{m\gamma f_0^2}{2[(\omega_0^2/\omega - \omega)^2 + \gamma^2]}$$

Hence, maximum power delivered (or dissipation) occurs at *power resonance frequency* $\omega = \omega_0$ because at this frequency the denominator becomes minimum to make the overall term maximum. In other words, when the frequency of driving force exactly matches with the natural frequency of the oscillator (i.e; $\omega_0$), power resonance occurs and as a result maximum power transfer occurs to the oscillator by the driving agency.

The frequency dependent variation of resonance curves are depicted in figure 4

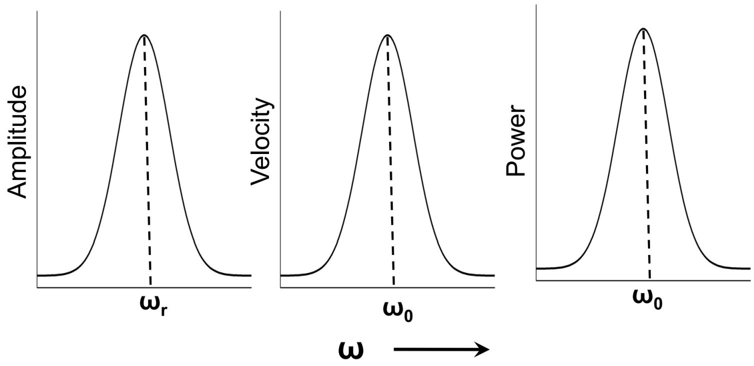


Figure 4: Resonance curves

#### 4.2.4 Sharpness of resonance

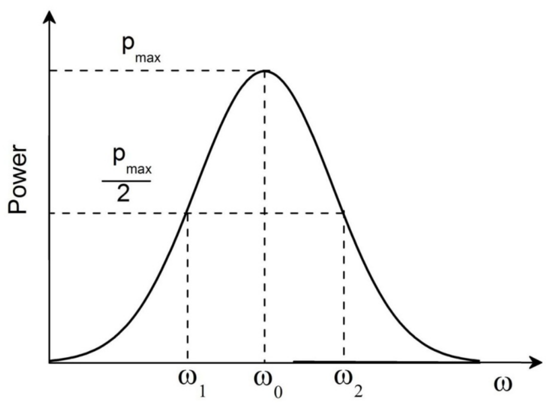


Figure 5: Power resonance curve.

Sharpness of resonance is a very important idea related to forced oscillation and it has great physical significance in designing receptor circuit. Maximum power absorption from external oscillatory force is attained at resonance frequency and away from this frequency power absorption gradually decreases. Therefore, if we have many driving sources having various driving frequencies then the system will absorb power from those driving system having frequencies close to natural frequency of the oscillator (i.e; $\omega_0$). That means if the power resonance curve of an oscillator is sharp then it only absorb power from very narrow frequency window and rejects all frequencies beyond this narrow window. Generally, the acceptance of frequency range by an oscillator is defined by *band width* : $\Delta\omega = (\omega_2 - \omega_1)$; where $\omega_2$ and $\omega_1$ are known as *half power frequencies.* At half power frequencies, the power becomes half of its maximum value (maximum value occurs at resonance).

Therefore, at half power frequencies $\langle p \rangle = p_{max}/2$.

From equation (18),

$$\langle p \rangle = \frac{m\gamma\omega^2 f_0^2}{2|Z|^2} = \frac{m\gamma f_0^2}{2[(\omega_0^2/\omega - \omega)^2 + \gamma^2]}$$

At resonance frequency $\omega = \omega_0$, we have $\langle p \rangle = p_{max}$. Therefore,

$$p_{max} = \frac{mf_0^2}{2\gamma}$$

Now at half power frequencies,

$$\begin{aligned} & \langle p \rangle = p_{max}/2 \\ \Rightarrow \quad & \frac{m\gamma f_0^2}{2[(\omega_0^2/\omega - \omega)^2 + \gamma^2]} = \frac{m f_0^2}{4\gamma} \\ \Rightarrow \quad & (\omega_0^2 - \omega^2)^2 = \omega^2\gamma^2 \\ \Rightarrow \quad & \omega_0^2 - \omega^2 = \pm\omega\gamma \\ \Rightarrow \quad & \omega^2 \pm \gamma\omega - \omega_0^2 = 0 \\ \Rightarrow \quad & \omega = \frac{1}{2}\left[\mp\gamma \pm \sqrt{\gamma^2 + 4\omega_0^2}\right] \end{aligned} \tag{20}$$

As expected, we have four solutions of $\omega$ as it was a forth order equation of $\omega$. However, only two of these solutions are physically acceptable;

$$\omega_1 = \frac{1}{2}\left[-\gamma + \sqrt{\gamma^2 + 4\omega_0^2}\right]$$

and

$$\omega_2 = \frac{1}{2}\left[\gamma + \sqrt{\gamma^2 + 4\omega_0^2}\right]$$

Other two solutions yield negative value of $\omega$ which is non-physical. These tow half power frequencies are depicted in the power resonance curve (figure 5). Hence the band width $\Delta\omega$ is defined as the difference between two half power frequencies,

$$\Delta\omega = \omega_2 - \omega_1 = \gamma$$

The *quality factor* $Q$ is defined as,

$$Q = \frac{\omega_0}{\Delta\omega}$$

According to definition of the quality factor it measures the sharpness of resonance curve or the selectivity of oscillating system. If $Q$ is large, then it implies that the curve is sharp and more selective.

## 5 Superposition of SHM

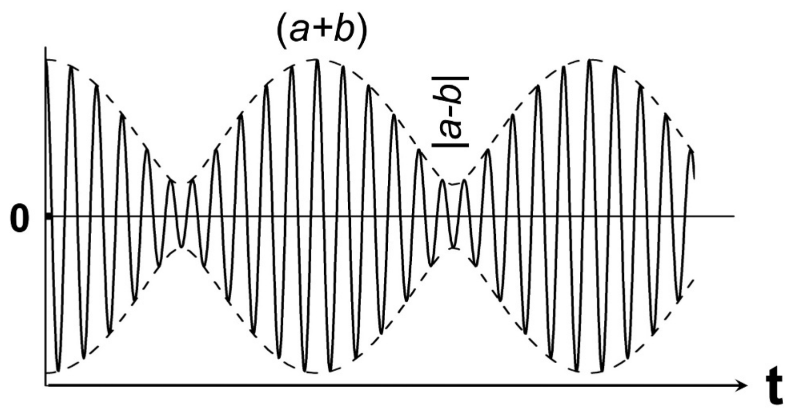


Figure 6: Beats : superposition of two co linear oscillations; $x_1(t) = a\cos(\omega_1 t)$ and $x_2(t) = b\cos(\omega_2 t)$ such that $\omega_1 \approx \omega_2$. In this figure, $\omega_1 = 10$ and $\omega_2 = 11$

# 6 Coupled oscillation

# 7 Waves

## 8 Group velocity and phase velocity

A pure sinusoidal wave of single frequency is infinitely extended in space-time. It neither has any starting point and nor a end point. It exhibits full space-time translational symmetry. Therefore, an information by means of wavy signal can not be propagated through pure sinusoidal wave. To flow a signal through wave, the shape of the wave should be changed somehow different from pure sinusoidal. The signal have to have a starting point and a end point in space-time; that means it has a finite extension in space time. Such a propagating wave is called *wave packet*. The wave packet propagates through medium and carries information. For example, switching on-off of a light source will produce a light wave (EM wave) packet of finite extension in space-time and this wave packet caries signal from one point to another point in space.

A propagating wave packet has many frequency components associated with pure sinusoidal waves. In other words, many sinusoidals having various wavelengths (or wave vectors) and amplitudes superpose to give rise wave packet. The shape and velocity of propagation of wave packet depends on the contributions of sinusoidals. Though the sinusoidal wave propagates with their characteristic velocity called *pahse velocity*: $v_p$; the wave packet propagates with somehow different velocity called *group velocity*: $v_g$. As the wave packet is basically group of many sinusoidal waves, thats why its velocity is known to be group velocity.

The mathematical form of pure sinusoidal wave propagating along $x$-axis is given by;

$$\xi(x,t) = a\exp[i(kx-\omega t)] \tag{21}$$

Here the wave vector $k$, angular frequency $\omega$ and phase velocity $v_p$ are connected as;

$$\omega = kv_p$$

In dispersive medium, $v_p$ is a function of $k$ that means $\lambda$. Therefore,

$$\omega(k) = kv_p(k)$$

The wave packet is superposition of various sinusoidal waves of various amplitudes and frequencies, therefore, mathematically a wave packet can be described as,

$$\Psi(x,t) = \sum_k \xi_k(x,t) = \sum_k a_k \exp[i(kx-\omega(k)t)] \tag{22}$$

$k$ is a continuous variable, therefore the discrete sum is to be transformed to integral as;

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \varphi(k)\exp[i(kx-\omega(k)t)]\ dk \tag{23}$$

The prefactor of $1/\sqrt{2\pi}$ is for convention.

At the starting time $t=0$, the wave packet is generated at some point $x$. The initial shape of wave packet is;

$$\Psi(x,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \varphi(k)\exp(ikx)\ dk \tag{24}$$

This is basically Fourier transformation of the function $\varphi(k)$. The function $\varphi(k)$ can be obtained by inverse Fourier transformation of $\Psi(x,0)$ as,

$$\varphi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \Psi(x,0)\exp(-ikx)\ dx \tag{25}$$

At time $t=0$, the wave packet $\Psi(x,0)$ is produced at point $x$ and then it propagates with velocity $v_g$ and reaches at some point $x' = x + v_g t$ after time $t$. So our aim is to find out $\Psi(x,t)$ using given shape of $\Psi(x,0)$ and to identify $v_g$.

Let us assume $\varphi(k)$ is sharply peaked function of $k$ about $k = k_0$. If we consider wide range of variation of $\varphi(k)$ over $k$ then we have to also consider wide variation of $\omega(k)$ i.e; $v_p(k)$ to integrate Eq.(23) (as they are related to each other). In such situation a large nos of sinusoidal waves having wide variation of $v_p(k)$ is superposing, therefore, the group of wave (wave packet) smears out too quickly. For sharply peaked $\varphi(k)$ about $k = k_0$, we can assume,

$$\omega(k) \approx \omega(k_o) + (k - k_0)\left(\frac{d\omega(k)}{dk}\right)_{k_0} = \omega_0 + (k - k_0)\omega_0'$$

Therefore, in Eq.(23) we substitute the approximate expression of $\omega(k)$,

$$\begin{aligned}
\Psi(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \varphi(k)\exp[i(kx - (\omega_0 + (k-k_0)\omega_0')t)]\ dk \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \varphi(k)\exp[i(kx - \omega_0 t - k\omega_0' t + k_0\omega_0' t)]\ dk \\
&= \exp[-i(\omega_0 - k_0\omega_0')t] \times \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \varphi(k)\exp[i(x - \omega_0' t)k]\ dk
\end{aligned} \tag{26}$$

Now we transform the variable $x \to x + \omega_0' t$

$$\begin{aligned}
\Psi(x + \omega_0' t, t) &= \exp[-i(\omega_0 - k_0\omega_0')t] \times \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \varphi(k)\exp(ikx)\ dk \\
&= \exp[-i(\omega_0 - k_0\omega_0')t]\ \Psi(x,0)
\end{aligned}$$

The last expression says that after starting of the wave packet $\Psi(x,0)$ at time $t = 0$ and at position $x$, it travels to a distance $x' = x + \omega_0' t$ after time $t$. The prefactor $\exp[-i(\omega_0 - k_0\omega_0')t]$ is a phase factor, it does not have any contribution to the intensity of wave packet. Here, one thing is clear that the wave packet $\Psi$ travels with a velocity of $v_g = \omega_0'$ according to the expression. That means we can express the group velocity and phase velocity as;

$$\boxed{v_g = \frac{d\omega(k)}{dk}} \quad \text{and} \quad \boxed{v_p = \frac{\omega(k)}{k}} \tag{27}$$

### 8.1 Relation between phase velocity and group velocity

$$\begin{aligned}
v_g &= \frac{d\omega(k)}{dk} \\
&= \frac{d}{dk}(kv_p(k)) \\
&= v_p(k) + k\frac{dv_p(k)}{dk} \\
&= v_p(\lambda) + \frac{2\pi}{\lambda}\frac{dv_p(\lambda)}{d\lambda}\frac{d\lambda}{dk} \qquad ; \qquad k = \frac{2\pi}{\lambda} \\
&= v_p(\lambda) + \frac{2\pi}{\lambda}\frac{dv_p(\lambda)}{d\lambda}\left(-\frac{\lambda^2}{2\pi}\right) \qquad ; \qquad \frac{d\lambda}{dk} = -\frac{\lambda^2}{2\pi} \\
&= v_p(\lambda) - \lambda\frac{dv_p(\lambda)}{d\lambda}
\end{aligned} \tag{28}$$

Therefore, relation between $v_g$ and $v_p$ is,

$$\boxed{v_g = v_p - \lambda\frac{dv_p}{d\lambda}} \tag{29}$$

## 9 Exercise with hint and solution

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