Numerical Methods

Dr. Phonindra Nath Das

Department of Mathematics
Ramakrishna Mission Vivekananda Centenary College

Email: *phonimath@gmail.com*

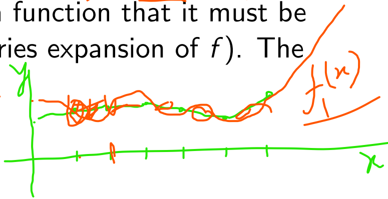
September 6, 2021

Outline of presentation

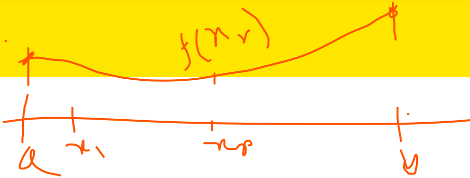
Truncation error:

Suppose that we want to compute $f(x)$ where x is a real number and f is a real function which we so far do not specify any closer. In practical computations the number x must be approximated by a rational number x' since no computer can store numbers with an infinite number of decimals.

The difference $x' - x$ constitutes the initial error while the difference $\epsilon = f(x') - f(x)$ is the corresponding propagated error. In many cases f is such a function that it must be replaced by a simpler function f_1, (often a truncated power series expansion of f). The difference $\epsilon_r = f_1(x') - f(x')$ is then the truncation error.



Finite Differences



Let $y = f(x)$ be specified by a given explicit expression defined in a closed interval $[a, b]$. Now some values of x are given then we can easilly obtain corresponding values of $f(x)$.

But the problem is when no such expression of $f(x)$ is given then we are unable to obtain the exact values of y even though x is given.

In that case we can approximate the values of the function up to a desired degrees of accuracy with the help of the virtue of **finite differences**, which deals with the values of x_i and corresponding functional values that is $f(x_i)$.

Forward differences

Let $y = f(x)$ be a function defined in $[a, b]$ and consider the consecutive values of x, each differs by h with previous or next of x_i such that $x_i - x_{i-1} = h$. Here, we have $a = x_0, x_1 = x_0 + h, x_2 = x_1 + h = x_0 + 2h, \cdots, x_r = x_0 + rh, \cdots, x_n = x_{n-1} + nh = b$. Let the corresponding functional values are respectively

$$\begin{array}{ccccc} y_0 & = & f(x_0) & = & f(x_0 + 0 \cdot h), \\ y_1 & = & f(x_1) & = & f(x_0 + h), \\ y_2 & = & f(x_2) & = & f(x_0 + 2h), \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ y_r & = & f(x_r) & = & f(x_0 + rh), \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ y_n & = & f(x_n) & = & f(x_0 + nh). \end{array}$$

Here the values of x which are $x_0, x_1, \cdots, x_n$, are called the *arguments* or *nodes* whereas the values of the function $y = f(x)$ which are $y_0, y_1, \cdots, y_n$, that is, the values of corresponding arguments are called the *entries*.

The differences $f(x_0+h)-f(x_0), f(x_0+2h)-f(x_0+h), \cdots, f(x_0+nh)$ - $f(x_0+(n-1)h)$ are called the first forward differences of $y=f(x)$ which are denoted by

$$\begin{array}{ccccccc} \Delta f(x_0) & = & f(x_0+h)-f(x_0) & = & y_1-y_0 & = & \Delta y_0, \\ \Delta f(x_0+h) & = & f(x_0+2h)-f(x_0+h) & = & y_2-y_1 & = & \Delta y_1, \\ \Delta f(x_0+2h) & = & f(x_0+3h)-f(x_0+2h) & = & y_3-y_2 & = & \Delta y_2, \\ \cdots & \cdots & \cdots \quad \cdots \quad \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

Thus the general form is

$$\Delta f(x_0+(n-1)h) = f(x_0+nh)-f(x_0+(n-1)h) = y_n-y_{n-1} = \Delta y_{n-1}.$$

The differences of the first forward differences are called the second forward differences.

Second forward differences are denoted as follows

$$
\begin{aligned}
\Delta^2 f(x_0) &= \Delta f(x_0+h) - \Delta f(x_0) \\
&= f(x_0+2h) - f(x_0+h) - [f(x_0+h) - f(x_0)] \\
&= f(x_0+2h) - 2f(x_0+h) + f(x_0) \\
&= y_2 - 2y_1 + y_0 \;=\; \Delta^2 y_0 \\
\Delta^2 f(x_0+h) &= \Delta f(x_0+2h) - \Delta f(x_0+h) \\
&= f(x_0+3h) - f(x_0+2h) - [f(x_0+2h) - f(x_0+h)] \\
&= f(x_0+3h) - 2f(x_0+2h) + f(x_0+h) \\
&= y_3 - 2y_2 + y_1 \;=\; \Delta^2 y_1 \\
\Delta^2 f(x_0+2h) &= \Delta f(x_0+3h) - \Delta f(x_0+2h) \\
&= f(x_0+4h) - f(x_0+3h) - [f(x_0+3h) - f(x_0+2h)] \\
&= f(x_0+4h) - 2f(x_0+3h) + f(x_0+2h) \\
&= y_4 - 2y_3 + y_2 \;=\; \Delta^2 y_2
\end{aligned}
$$

... ...

Similarly the third forward differences are

$$\begin{aligned}
\Delta^3 f(x_0) &= \Delta^2 f(x_0+h) - \Delta^2 f(x_0) \\
&= f(x_0+3h) - 2f(x_0+2h) + f(x_0+h) \\
&\quad - [f(x_0+2h) - 2f(x_0+h) + f(x_0)] \\
&= f(x_0+3h) - 3f(x_0+2h) + 3f(x_0+h) - f(x_0) \\
&= y_3 - 3y_2 + 3y_1 - y_0 \; = \; \Delta^3 y_0 \\
\Delta^3 f(x_0+h) &= \Delta^2 f(x_0+2h) - \Delta^2 f(x_0+h) \\
&= f(x_0+4h) - 2f(x_0+3h) + f(x_0+2h) \\
&\quad - [f(x_0+3h) - 2f(x_0+2h) + f(x_0+h)] \\
&= f(x_0+4h) - 3f(x_0+3h) + 3f(x_0+2h) - f(x_1) \\
&= y_4 - 3y_3 + 3y_2 - y_1 \; = \; \Delta^3 y_1
\end{aligned}$$

and so on.

Backward Differences

The differences
$f(x_0+h)-f(x_0), f(x_0+2h)-f(x_0+h), \cdots, f(x_0+nh)-F(x_0+(n-1)h)$ are also sometimes called the first backward differences of $y=f(x)$ which are denoted by

$$\begin{array}{ccccccc} \nabla f(x_0+h) & = & f(x_0+h)-f(x_0) & = & y_1-y_0 & = & \nabla y_1, \\ \nabla f(x_0+2h) & = & f(x_0+2h)-f(x_0+h) & = & y_2-y_1 & = & \nabla y_2, \\ \nabla f(x_0+3h) & = & f(x_0+3h)-f(x_0+2h) & = & y_3-y_2 & = & \nabla y_3, \\ \cdots & \cdots & \cdots \quad \cdots \quad \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

Thus the general form is

$$\nabla f(x_0+nh)$$
$$= f(x_0+nh)-f(x_0+(n-1)h) \;=\; y_n-y_{n-1} \;=\; \nabla y_n.$$

The differences of the first backward differences are called the second backward differences.

Second backward differences are denoted as

$$
\begin{aligned}
\nabla^2 f(x_0+2h) &= \Delta f(x_0+2h) - \nabla f(x_0+h) \\
&= f(x_0+2h) - f(x_0+h) - [f(x_0+h) - f(x_0)] \\
&= f(x_0+2h) - 2f(x_0+h) + f(x_0) \\
&= y_2 - 2y_1 + y_0 \;=\; \nabla^2 y_2 \\
\nabla^2 f(x_0+3h) &= \nabla f(x_0+3h) - \nabla f(x_0+2h) \\
&= f(x_0+3h) - f(x_0+2h) - [f(x_0+2h) - f(x_0+h)] \\
&= f(x_0+3h) - 2f(x_0+2h) + f(x_0+h) \\
&= y_3 - 2y_2 + y_1 \;=\; \nabla^2 y_3 \\
\nabla^2 f(x_0+4h) &= \nabla f(x_0+4h) - \nabla f(x_0+3h) \\
&= f(x_0+4h) - f(x_0+3h) - [f(x_0+3h) - f(x_0+2h)] \\
&= f(x_0+4h) - 2f(x_0+3h) + f(x_0+2h) \\
&= y_4 - 2y_3 + y_2 \;=\; \nabla^2 y_4
\end{aligned}
$$

$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$

Similarly the third backward differences are

$$\begin{aligned}
\nabla^3 f(x_0+3h) &= \nabla^2 f(x_0+3h) - \nabla^2 f(x_0+2h) \\
&= f(x_0+3h) - 2f(x_0+2h) + f(x_0+h) \\
&\quad - [f(x_0+2h) - 2f(x_0+h) + f(x_0)] \\
&= f(x_0+3h) - 3f(x_0+2h) + 3f(x_0+h) - f(x_0) \\
&= y_3 - 3y_2 + 3y_1 - y_0 \;=\; \nabla^3 y_3 \\
\nabla^3 f(x_0+4h) &= \nabla^2 f(x_0+4h) - \nabla^2 f(x_0+3h) \\
&= f(x_0+4h) - 2f(x_0+3h) + f(x_0+2h) \\
&\quad - [f(x_0+3h) - 2f(x_0+2h) + f(x_0+h)] \\
&= f(x_0+4h) - 3f(x_0+3h) + 3f(x_0+2h) - f(x_1) \\
&= y_4 - 3y_3 + 3y_2 - y_1 \;=\; \nabla^3 y_4
\end{aligned}$$

and so on.

Shift operator

Let $y = f(x)$ be an arbitrary function and h be a non zero constant, then an operator E defined by

$$Ef(x) = f(x+h) \tag{1}$$

is called the shifting operator or shift operator or displacement operator

The higher order shift operators are defined as follows

$$\begin{aligned} E^2 f(x) &= E \cdot Ef(x) = Ef(x+h) = f(x+2h) \\ E^3 f(x) &= E \cdot Ef(x+h) = Ef(x+2h) = f(x+3h). \end{aligned}$$

In general we have

$$E^n f(x) = f(x+nh) \tag{2}$$

$$E^{-n} f(x) = f(x-nh) \tag{3}$$

Note: It should always remember that $E^n f(x) \neq \left[Ef(x)\right]^n$ and $E^{-n} f(x) \neq \frac{1}{\left[Ef(x)\right]^n}$.

Difference Table

Based of the different operators we construct a table to easily compute various numerical problem and such a table is called the *difference table*.

For both forward and backward difference operator we can construct two types of difference table namely *Diagonal forward difference table* and *Horizontal forward difference table*.

In Diagonal forward difference table the differences are placed at the middle position between two nodes. From such a table one can easily identified the forward differences for a particular node.

For example in the Table (see next slide) forward differences of x_1 are placed on the forward diagonal position passing through the node x_1 and highlighted by underlined mark.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\cdots$	$\Delta^n y$
x_0	y_0					
		Δy_0				
x_1	$\underline{y_1}$		$\Delta^2 y_0$			
		$\underline{\Delta y_1}$		$\Delta^3 y_0$		
x_2	y_2		$\underline{\Delta^2 y_1}$			$\Delta^n y_0$
		Δy_2				
x_3	y_3		$\cdots$			
$\cdots$	$\cdots$	$\cdots$	$\Delta^2 y_{n-2}$			
		Δy_{n-1}				
x_n	y_n					

Table: Diagonal forward difference table.

Now we provide the horizontal forward difference table by the following way.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\cdots$	$\Delta^{n-1} y$	$\Delta^n y$
x_0	y_0	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\cdots$	$\Delta^{n-1} y_0$	$\Delta^n y_0$
x_1	y_1	Δy_1	$\Delta^2 y_1$	$\Delta^3 y_1$	$\cdots$	$\Delta^{n-1} y_1$	
x_2	y_2	Δy_2	$\Delta^2 y_2$	$\Delta^3 y_2$	$\cdots$		
x_3	y_3	Δy_3	$\Delta^2 y_3$	$\Delta^3 y_3$	$\cdots$		
$\vdots$		$\vdots$		$\vdots$			
x_{n-2}	y_{n-2}	Δy_{n-2}	$\Delta^2 y_{n-2}$				
x_{n-1}	y_{n-1}	Δy_{n-1}					
x_n	y_n						

Table: Horizontal forward difference table.

Similar to the forward difference table we construct two types of difference table namely Diagonal forward difference table and *Horizontal forward difference table* provide respectively in Tables next to this slide.

To get backward differences for a node say, x_n, its backward differences are placed on the backward diagonal position passing through the node x_n and highlighted by undelined mark in the Table of next slide.

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\cdots$	$\nabla^n y$
x_0	y_0					
		∇y_1				
			$\nabla^2 y_2$			
$\cdots$	$\cdots$					
x_{n-3}	y_{n-3}		$\cdots$		$\cdots$	$\underline{\nabla^n y_n}$
		∇y_{n-2}				
x_{n-2}	y_{n-2}		$\nabla^2 y_{n-1}$			
		∇y_{n-1}		$\underline{\nabla^3 y_n}$		
x_{n-1}	y_{n-1}		$\underline{\nabla^2 y_n}$			
		$\underline{\nabla y_n}$				
x_n	$\underline{y_n}$					

Table: Diagonal backward difference table.

Now we provide the horizontal backward difference table as:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\cdots$	$\nabla^{n-1} y$	$\nabla^n y$
x_0	y_0						
x_1	y_1	∇y_1					
x_2	y_2	∇y_2	$\nabla^2 y_2$				
x_3	y_3	∇y_3	$\nabla^2 y_3$	$\nabla^3 y_3$			
$\vdots$		$\vdots$		$\vdots$			
x_{n-2}	y_{n-2}	∇y_{n-2}	$\nabla^2 y_{n-2}$	$\nabla^3 y_{n-2}$	$\cdots$		
x_{n-1}	y_{n-1}	∇y_{n-1}	$\nabla^2 y_{n-1}$	$\nabla^3 y_{n-1}$	$\cdots$	$\nabla^{n-1} y_{n-1}$	
x_n	y_n	∇y_n	$\nabla^2 y_n$	$\nabla^3 y_n$	$\cdots$	$\nabla^{n-1} y_n$	$\nabla^n y_n$

Table: Horizontal backward difference table.