Numerical Methods and Numerical Methods Lab

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Outline of presentation

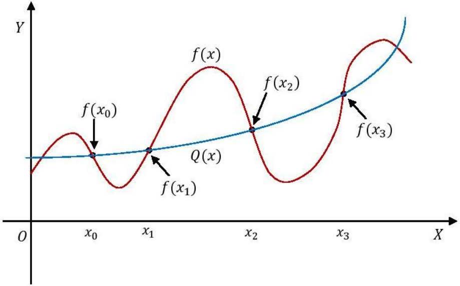
Interpolation

Imagine that there is an unknown function $f(x)$ for which someone supplies you with its (exact) values at $(n+1)$ distinct points $x_0 < x_1 < \cdots < x_n$, i.e., $f(x_0), f(x_1), \cdots, f(x_n)$ are given. The interpolation problem is to construct a function $Q(x)$ that passes through these points, i.e., to find a function $Q(x)$ such that the interpolation requirements

$$Q(x_i) = f(x_i), \quad 0 \leq i \leq n, \tag{1}$$

are satisfied. One easy way of obtaining such a function, is to connect the given points with straight lines. While this is a legitimate solution of the interpolation problem, usually (though not always) we are interested in a different kind of a solution, e.g., a smoother function. We therefore always specify a certain class of functions from which we would like to find one that solves the interpolation problem.

For example, we may look for a function $Q(x)$ that is a polynomial, $Q(x)$. Alternatively, the function $Q(x)$ can be a trigonometric function or a piecewise-smooth polynomial, and so on.



As a simple example let us consider values of a function that are prescribed at two points: $(x_0, f(x_0))$ and $(x_1, f(x_1))$. There are infinitely many functions that pass through these two points. However, if we limit ourselves to polynomials of degree less than or equal to one, there is only one such function that passes through these two points: the line that connects them. A line, in general, is a polynomial of degree one, but if the two given values are equal, $f(x_0) = f(x_1)$, the line that connects them is the constant $Q(x_0) \equiv f(x_0)$, which is a polynomial of degree zero.

This is why we say that there is a unique polynomial of degree ≤ 1 that connects these two points. The points $x_0, x_1, \cdots, x_n$ are called the interpolation points. The property of passing through these points is referred to as interpolating the data. The function that interpolates the data is an interpolant or an interpolating polynomial (or whatever function is being used). There are cases were the interpolation problem has no solution, e.g., if we look for a linear polynomial that interpolates three points that do not lie on a straight line. When a solution exists, it can be unique (a linear polynomial and two points), or the problem can have more than one solution (a quadratic polynomial and two points). What we are going to study in this section is precisely how to distinguish between these cases. We are also going to present different approaches to constructing the interpolant. Other than agreeing at the interpolation points, the interpolant $Q(x)$ and the underlying function $f(x)$ are generally different. The interpolation error is a measure on how different these two functions are. We will study ways of estimating the interpolation error. We will also discuss strategies on how to minimize this error.

It is important to note that it is possible to formulate the interpolation problem without referring to any underlying function $f(x)$. For example, we may have a list of interpolation points $x_0, x_1, \cdots, x_n$ and data that is experimentally collected at these points, $y_0, y_1, \cdots, y_n$ which we would like to interpolate. The solution to this interpolation problem is identical to the one where the values are taken from an underlying function.

Newton's foreword interpolation

Let $y = f(x)$ in $[a, b]$ and is known on $(n+1)$ arguments $x_0, x_1, \cdots, x_n$, equally spaced in $[a, b]$. Now we want to find a polynomial $L_n(x)$ of degree $\leq n$ such that it replaces $f(x)$ on the set of interpolating points x_i $(i = 0, 1, \cdots, n)$, so

$$L_n(x_i) = f(x_i) = y_i, \quad i = 0, 1, \cdots, n. \tag{2}$$

Let us take the interpolation polynomial $L_n(x)$ in form of

$$\begin{aligned} L_n(x) &= C_0 + C_1(x - x_0) + C_2(x - x_0)(x - x_1) + \cdots \\ &\quad + C_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}) \end{aligned}$$

$C_0, C_1, \cdots, C_n$ are constants to be determined by using (2). Differences of $L_n(x)$ gives

$$\begin{aligned} \Delta L_n(x) &= hC_1 + 2hC_2(x - x_0) + \cdots + nhC_n(x - x_0)\cdots(x - x_{n-2}) \\ \Delta^2 L_n(x) &= 2.1h^2 C_2 + 3.2h^2 C_3(x - x_0) + \cdots \\ &\quad + n(n-1)h^2 C_n(x - x_0)(x - x_1)\cdots(x - x_{n-3}) \\ \cdots & \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ \Delta^n L_n(x) &= n!h^n C_n \end{aligned}$$

Now putting $x = x_0$ in the above relations and using (2) we get

$$\begin{aligned} C_0 &= L_n(x_0) = f(x_0) \\ C_1 &= \frac{\Delta L_n(x_0)}{h} = \frac{\Delta f(x_0)}{h} \\ C_2 &= \frac{\Delta^2 L_n(x_0)}{2!h^2} = \frac{\Delta^2 f(x_0)}{2!h^2} \\ \cdots & \quad \cdots \quad \cdots \\ C_n &= \frac{\Delta^n L_n(x_0)}{n!h^n} = \frac{\Delta^n f(x_0)}{n!h^n} \end{aligned}$$

Hence

$$\begin{aligned} L_n(x) &= f(x_0) + \frac{(x-x_0)}{1!h}\Delta f(x_0) + \frac{(x-x_0)(x-x_1)}{2!h^2}\Delta^2 f(x_0) + \\ &\cdots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{n!h^n}\Delta^n f(x_0) \end{aligned}$$

Let $u = \frac{x-x_0}{h}$, clearly u is a dimensionless quantity called the *phase* and generally we have $u - i = \frac{x-x_i}{h}$. Then the above expression becomes

$$L_n(x) = f(x_0) + \frac{u}{1!}\Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0) + \cdots + \frac{u(u-1)\cdots(u-n+1)}{n!}\Delta^n f(x_0)$$

And the remainder is given by

$$R_{n+1}(x) = \frac{u(u-1)\cdots(u-n)}{(n+1)!}h^{n+1}f^{(n+1)}(\xi),\ x_0 < \xi < x_n \tag{3}$$

Hence, *Newton's forward interpolation formula* with corresponding remainder may be written as

$$f(x) = f(x_0) + \binom{u}{1}\Delta f(x_0) + \binom{u}{2}\Delta^2 f(x_0) + \cdots + \binom{u}{n}\Delta^n f(x_0) + \binom{u}{n+1}h^{n+1}f^{(n+1)}(\xi),\ x_0 < \xi < x_n \tag{4}$$

In term of y the formula can be written as

$$y = y_0 + \binom{u}{1}\Delta y_0 + \binom{u}{2}\Delta^2 y_0 + \cdots + \binom{u}{n}\Delta^n y_0 + R_{n+1}(x)$$

The point x_0 is called the starting point of the formula.

A difference table have the following form to compute Newton's forward interpolation formula, where the differences used in the formula are highlighted by underline.

x	y	Δy	$\Delta^2 y$	$\Delta^n y$
x_0	$\underline{y_0}$			
		$\underline{\Delta y_0}$		
x_1	y_1		$\underline{\Delta^2 y_0}$	
		Δy_1		
x_2	y_2		$\cdots$	$\underline{\Delta^n y_0}$
$\cdots$	$\cdots$	$\cdots$	$\Delta^2 y_{n-2}$	
		Δy_{n-1}		
x_n	y_n			

Table: Difference table for Newton's forward interpolation formula.

Newton's backward interpolation

Newton's backward interpolation formula is used for interpolating $f(x)$ for values of x lying at the end of the table. In this case we arrange the arguments in reverse order and the interpolating polynomial $L_n(x)$ is considered as

$$\begin{aligned} L_n(x) &= C_0 + C_1(x - x_n) + C_2(x - x_n)(x - x_{n-1}) + \cdots \\ &\quad + C_n(x - x_n)(x - x_{n-1}) \cdots (x - x_1) \end{aligned}$$

Now taking succesive difference we obtain

$$\begin{aligned} \Delta L_n(x) &= hC_1 + 2hC_2(x - x_{n-1}) + \cdots + nhC_n(x - x_{n-1}) \cdots (x - x_1) \\ \Delta^2 L_n(x) &= 2.1h^2 C_2 + 3.2h^2 C_3(x - x_{n-2}) + \cdots \\ &\quad + n(n-1)h^2 C_n(x - x_{n-2}) \cdots (x - x_1) \\ \cdots & \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ \Delta^n L_n(x) &= n! h^n C_n \end{aligned}$$

Now putting $x = x_n, x_{n-1}, \cdots, x_1, x_0$ in the above relations respectively and using (2) we get

$$\begin{aligned} C_0 &= L_n(x_n) = f(x_n) \\ C_1 &= \frac{\Delta L_n(x_{n-1})}{h} = \frac{\Delta f(x_{n-1})}{h} \\ C_2 &= \frac{\Delta^2 L_n(x_{n-2})}{2!h^2} = \frac{\Delta^2 f(x_{n-2})}{2!h^2} \\ \cdots & \quad \cdots \quad \cdots \\ C_n &= \frac{\Delta^n L_n(x_0)}{n!h^n} = \frac{\Delta^n f(x_0)}{n!h^n} \end{aligned}$$

which gives

$$\begin{aligned} L_n(x) &= f(x_n) + \frac{(x-x_n)}{1!h}\Delta f(x_{n-1}) + \frac{(x-x_n)(x-x_{n-1})}{2!h^2}\Delta^2 f(x_{n-2}) + \\ & \cdots + \frac{(x-x_n)(x-x_{n-1})\cdots(x-x_1)}{n!h^n}\Delta^n f(x_0) \end{aligned}$$

Here the phase is defined by $u = \frac{x-x_n}{h}$ and noting that $u + i = \frac{x-x_{n-i}}{h}$ we obtain

$$L_n(x) = f(x_n) + \frac{u}{1!}\Delta f(x_{n-1}) + \frac{u(u+1)}{2!}\Delta^2 f(x_{n-2}) + \cdots + \frac{u(u+1)\cdots(u+n-1)}{n!}\Delta^n f(x_0)$$

and the remainder term is

$$R_{n+1}(x) = \frac{u(u+1)\cdots(u+n)}{(n+1)!}h^{n+1}f^{(n+1)}(\xi), \;\; x_0 < \xi < x_n \tag{5}$$

Hence *Newton's backward interpolation formula* is

$$f(x) = f(x_n) + \binom{u}{1}\Delta f(x_{n-1}) + \binom{u+1}{2}\Delta^2 f(x_{n-2}) + \cdots + \binom{u+n-1}{n}\Delta^n f(x_0) + \binom{u+n}{n+1}h^{n+1}f^{(n+1)}(\xi), \quad x_0 < \xi < x_n \tag{6}$$

In term of y the formula can be obtain as in forward case. Here x_n is the starting point. The coefficients of backward formula can linked with forward case as follows

$$\binom{u+i-1}{i} = (-1)^i \binom{-u}{i} \tag{7}$$

so that using coefficients from forward we can obtain coefficients for backward formula. A difference table have the following form to compute Newton's backward interpolation formula, where the differences used in the formula are highlighted by underline.

x	y	Δy	$\Delta^2 y$	$\Delta^n y$
x_0	y_0			
		Δy_0		
x_1	y_1		$\Delta^2 y_0$	
...	...	...	...	
		Δy_1		
x_{n-2}	y_{n-2}			$\underline{\Delta^n y_0}$
		Δy_{n-2}		
x_{n-1}	y_{n-1}		$\underline{\Delta y_{n-2}}$	
		$\underline{\Delta y_{n-1}}$		
x_n	$\underline{y_n}$			

Example 1:
Find the values of $f(1.5)$ and $f(7.5)$ using the following table:

x	:	1	2	3	4	5	6	7	8
f(x)	:	1	8	27	64	125	216	343	512

Note that the given nodes are equispaced and we have to compute the functional value near to the begining $(x = 1.5)$ and end $(x = 7.5)$ of the nodes. So, we can uuse Nowton's forroward and backward interpolation formula. Let us construct the difference table as follows:

x	$y=f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	1			
		7		
2	8		12	
		19		6
3	27		18	
		37		6
4	64		24	
		61		6
5	125		30	
		91		6
6	216		36	
		127		6
7	343		42	
		169		
8	512			

(i) To compute $f(1.5)$, we use Newton's forward interpolation formula (4), as the point $x = 1.5$ is near to the begining of the above table. Therefore, we have

$$\begin{aligned} f(x) &= f(x_0) + \binom{u}{1}\Delta f(x_0) + \binom{u}{2}\Delta^2 f(x_0) + \binom{u}{3}\Delta^3 f(x_0) + \cdots \\ &= f(x_0) + u \cdot \Delta f(x_0) + u(u-1)\cdot\frac{\Delta^2 f(x_0)}{2!} + u(u-1)(u-2)\cdot\frac{\Delta^3 f(x_0)}{3!} + \cdots \end{aligned}$$

where $u = \frac{x-x_0}{h}$. Here $x = 1.5$, $x_0 = 1$ and $h = 1$. So, $u = \frac{1.5-1}{1} = 0.5$.
Therefore,

$$\begin{aligned} f(1.5) &= 1 + 0.5 \times 7 + 0.5(-0.5) \times \frac{12}{2!} + 0.5(-0.5)(-1.5) \times \frac{6}{3!} \\ &= 1 + 3.5 - 1.5 + 0.375 \\ &= 3.375. \end{aligned}$$

(ii) To compute $f(7.5)$, we use Newton's backward interpolation formula (6), as the point $x = 7.5$ is near to the ending of the difference table. Therefore, we have

$$\begin{aligned} f(x) &= f(x_n) + \binom{u}{1}\Delta f(x_{n-1}) + \binom{u+1}{2}\Delta^2 f(x_{n-2}) + \binom{u+2}{3}\Delta^3 f(x_{n-3}) + \cdots \\ &= f(x_n) + u \cdot \Delta f(x_{n-1}) + u(u+1) \cdot \frac{\Delta^2 f(x_{n-2})}{2!} + u(u+1)(u+2) \cdot \frac{\Delta^3 f(x_{n-3})}{3!} + \cdots \end{aligned}$$

where $u = \frac{x-x_n}{h}$. Here $x = 7.5$, $x_n = 8$ and $h = 1$. So, $u = \frac{7.5-8}{1} = -0.5$.
Therefore,

$$\begin{aligned} f(7.5) &= 512 + (-0.5) \times 169 + (-0.5)(0.5) \times \frac{42}{2!} + (-0.5)(0.5)(1.5) \times \frac{6}{3!} \\ &= 512 - 84.5 - 5.25 - 0.375 \\ &= 421.875. \end{aligned}$$