Numerical Methods

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Outline of presentation

1. Gauss Jordan method
2. Iterative methods

Gauss Jordan method

In this section, we learn to solve systems of linear equations using a process called the Gauss-Jordan method. The process begins by first expressing the system as a matrix, and then reducing it to an equivalent system by simple row operations. The process is continued until the solution is obvious from the matrix. The matrix that represents the system is called the augmented matrix, and the arithmetic manipulation that is used to move from a system to a reduced equivalent system is called a row operation.

Suppose we have the following system of linear equations

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn}x_n & = & b_n \end{array} \tag{1}$$

where $b_1, b_2, \cdots, b_m$ and a_{ij}, $1 \leq i \leq m$, $1 \leq j \leq n$ are given real numbers. First we write corresponding Augemented matrix as

$$A_g = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right]$$

b

A

Then we interchange rows if necessary to obtain a non-zero number in the first row, first column.
Now we follow the next two steps such a way that, use a row operation to get a 1 as the entry in the first row and first column and immediately use row operations to make all other entries as zeros in column one. This leads to A_g as follows-

$$A_g = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{bmatrix} \equiv \begin{bmatrix} 1 & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} & b_1^{(1)} \\ 0 & a_{22}^{(1)} & \cdots & a_{2n}^{(1)} & b_2^{(1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & a_{n2}^{(1)} & \cdots & a_{nn}^{(1)} & b_n^{(1)} \end{bmatrix}$$

Then, we interchange rows if necessary to obtain a nonzero number in the second row, second column. Use a row operation to make this entry 1. Use row operations to make all other entries as zeros in column two.

Which gives the next equivalent augmented matrix as-

$$A_g \equiv \begin{bmatrix} 1 & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} & b_1^{(1)} \\ 0 & a_{22}^{(1)} & \cdots & a_{2n}^{(1)} & b_2^{(1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & a_{n2}^{(1)} & \cdots & a_{nn}^{(1)} & b_n^{(1)} \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 & \cdots & a_{1n}^{(2)} & b_1^{(2)} \\ 0 & 1 & \cdots & a_{2n}^{(2)} & b_2^{(2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{nn}^{(2)} & b_n^{(2)} \end{bmatrix}$$

Repeating these steps, moving along the main diagonal until we reach the last row, or until the number is zero and finally we obtain an equivalent augemnted matrix as follows:

$$A_g \equiv \begin{bmatrix} 1 & 0 & \cdots & 0 & b_1^{(n)} \\ 0 & 1 & \cdots & 0 & b_2^{(n)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & b_n^{(n)} \end{bmatrix}$$

$\Rightarrow Ax = b^{(n)}$

The final matrix is called the reduced row-echelon form.

Corresponding solution is $x_1 = b_1^{(n)}$, $x_2 = b_2^{(n)}, \cdots, x_n = b_n^{(n)}$.

Example:

Solve the following system by the Gauss-Jordan method.

$$\begin{array}{ccccccc} 2x & + & y & + & 2z & = & 10 \\ x & + & 2y & + & z & = & 8 \\ 3x & + & y & - & z & = & 2 \end{array}$$

Solution: Corresponding augmented matrix is

$$\begin{bmatrix} 2 & 1 & 2 & 10 \\ 1 & 2 & 1 & 8 \\ 3 & 1 & -1 & 2 \end{bmatrix} \rightarrow (R_1 \rightarrow R_2) \rightarrow \begin{bmatrix} 1 & 2 & 1 & 8 \\ 2 & 1 & 2 & 10 \\ 3 & 1 & -1 & 2 \end{bmatrix}$$

$$\rightarrow (R_2 - 2R_1) \rightarrow \begin{bmatrix} 1 & 2 & 1 & 8 \\ 0 & -3 & 0 & -6 \\ 3 & 1 & -1 & 2 \end{bmatrix} \rightarrow (R_3 - 3R_1) \rightarrow \begin{bmatrix} 1 & 2 & 1 & 8 \\ 0 & -3 & 0 & -6 \\ 0 & -5 & -4 & -22 \end{bmatrix} \rightarrow$$

$$(R_2/(-3)) \rightarrow \begin{bmatrix} 1 & 2 & 1 & 8 \\ 0 & 1 & 0 & 2 \\ 0 & -5 & -4 & -22 \end{bmatrix} \rightarrow (\frac{R_3 + 5R_2}{R_1 - 2R_2}) \rightarrow \begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -4 & -12 \end{bmatrix} \rightarrow$$

$$(R_3/(-4)) \rightarrow \begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \rightarrow (R_1 - R_3) \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Clearly, the solution reads $x = 1$, $y = 2$ and $z = 3$.

Gauss Jacobi method

The first iterative technique is called the Gauss Jacobi method or only Jacobi method which consists of two major assumptions

i). The system

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & \vdots & \vdots & & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn}x_n & = & b_n \end{array} \qquad (2)$$

has unique solution.

ii). The principle diagonal elements in the coefficient matrix are non zero.

The methods: To begin, we solve the 1st equation for x_1, the 2nd equation for x_2 and so on to obtain the rewritten equations:

$$x_1 = \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3 - \cdots a_{1n}x_n)$$

$$x_2 = \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}x_3 - \cdots a_{2n}x_n)$$

$$\vdots$$

$$x_n = \frac{1}{a_{nn}}(b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots a_{n,n-1}x_{n-1})$$

Then make an initial guess of the solution $x^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, \cdots, x_n^{(0)})$. Substitute these values into the right hand side the of the rewritten equations to obtain the first approximation, $(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \cdots, x_n^{(1)})$. Which accomplishes the first iteration.

In the same way, the second approximation $\left(x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, \cdots, x_n^{(2)}\right)$ is computed by substituting the first approximation's x-vales into the right hand side of the rewritten equations.

By repeated iterations, we form a sequence of approximations

$$x^{(k)} = (x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, \cdots, x_n^{(k)}), \quad k = 1, 2, 3, \cdots$$

Therefore, for each generate the components $x_i^{(k)}$ of $x^{(k)}$ from $x^{(k-1)}$ by

$$x_i^{(k)} = \frac{1}{a_{ii}}\Big[\sum_{j=1, j\neq i}^{n}\big(-a_{ij}x_j^{(k-1)}\big) + b_i\Big], \quad \text{for } i = 1, 2, 3, \cdots, n.$$

Example:

Apply the Gauss Jacobi method to solve the following system of equations

$$\begin{aligned} 5x_1 - 2x_2 + 3x_3 &= -1 \\ -3x_1 + 9x_2 + x_3 &= 2 \\ 2x_1 - x_2 - 7x_3 &= 3 \end{aligned}$$

Continue iterations until two successive approximations are identical when rounded to three significant digits.

Solution: To begin, rewrite the system

$$\begin{aligned} x_1 &= \frac{-1}{5} + \frac{2}{5}x_2 - \frac{3}{5}x_3 \\ x_2 &= \frac{2}{9} + \frac{3}{9}x_1 - \frac{1}{9}x_3 \\ x_3 &= -\frac{3}{7} + \frac{2}{7}x_1 - \frac{1}{7}x_2 \end{aligned}$$

Choose the initial guess $x_1 = 0, x_2 = 0, x_3 = 0$

The first approximation is

$$x_1^{(1)} = \frac{-1}{5} + \frac{2}{5}(0) - \frac{3}{5}(0) = -0.200$$

$$x_2^{(1)} = \frac{2}{9} + \frac{3}{9}(0) - \frac{1}{9}(0) = 0.222$$

$$x_3^{(1)} = -\frac{3}{7} + \frac{2}{7}(0) - \frac{1}{7}(0) = -0.429$$

Continue iteration, we obtain

n	$k = 0$	$k = 1$	$k = 2$	$k = 3$	....	$k = 8$
$x_1^{(k)}$	0.000	-0.200	0.146	0.192	....	
$x_2^{(k)}$	0.000	0.222	0.203	0.328	....	
$x_2^{(k)}$	0.000	-0.429	-0.517	-0.416	....	

You have to find the rest of the iterative values absent in the table.