Numerical Methods

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System of linear equations

We consider the problem of finding numerically n scalars $x_1, x_2, ..., x_n$ which satisfy the conditions

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn}x_n & = & b_n \end{array} \qquad (1)$$

$A = [a_{ij}]_{n \times n}$

where $b_1, b_2, \cdots, b_m$ and a_{ij}, $1 \leq i \leq m$, $1 \leq j \leq n$ are given real numbers. The system (1) is known as a **system of n linear equation in n unknowns**.

n

We can write the system (1) as follows:

$$Ax = B$$

Where, $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$. A is called the **coefficients matrix**.

$AX = B$

By the virtue of linear algebra it is well known that the above system has a unique solution if and only if A^{-1} exists and this is true if and only if $\det A \neq 0$.

Now in the entire chapter we assume that $\det A \neq 0$ so that unique solution exists, then our aim will be to compute the n unknown vector x up to a desired degree of accuracy.

Broadly there are two types numerical methods are available to compute such system, namely:

The **direct or exact methods**, such as **Cramer's rule, Gauss's elimination method** etc.

The **iterative methods** such as **Gauss Jacobi method, Gauss Seidel method** etc.

$x = ?$

$y = ?$

Gauss's elimination method (Direct)

Let us assume that the system (1) has unique solution and consider the augmented matrix $[A|b]$ of the system (1).

$n \times n+1$

$Ax = B \quad x = \frac{B}{A}$

Now, using elementary row transformations, Gauss elimination method reduces the matrix A in the augmented matrix to an upper triangular form, such that $[A|b] \longrightarrow [U|c]$. Then by back substitution we obtain the solution. To do this we proceed as follows.

$|A \; b|$

First, we write

$$a_{ij}^{(1)} = a_{ij}, \quad b_i^{(1)} = b_i, \qquad i, j = 1, 2, \cdots, n \tag{2}$$

Let $a_{11}^{(1)} \neq 0$, multiply the first equation of (1) by $m_{i1} = -a_{i1}^{(1)}/a_{11}^{(1)}$ and then add to the ith equation when x_1 is eliminated from that equation, where $i = 2, 3, \cdots, n$, which gives the following equivalent system of equations

$$
\begin{aligned}
a_{11}^{(1)}x_1 \quad + \quad a_{12}^{(1)}x_2 \quad + \quad \cdots \quad + \quad a_{1n}^{(1)}x_n \quad &= \quad b_1^{(1)} \\
a_{22}^{(2)}x_2 \quad + \quad \cdots \quad + \quad a_{2n}^{(2)}x_n \quad &= \quad b_2^{(2)} \\
\cdots \qquad \cdots \qquad \cdots \quad &\quad\ \ \cdots \\
a_{n2}^{(2)}x_2 \quad + \quad \cdots \quad + \quad a_{nn}^{(2)}x_n \quad &= \quad b_n^{(2)}
\end{aligned}
\tag{3}
$$

where

$$
a_{ij}^{(2)} = a_{ij}^{(1)} + m_{i1}a_{1j}, \; b_i^{(2)} = b_i^{(1)} + m_{i1}b_1^{(1)}, \quad i, j = 2, 3, \cdots, n. \tag{4}
$$

Again assume, $a_{22}^{(2)} \neq 0$. We note that the set of equation (3) except the first one is a system of $n-1$ linear equations in the $n-1$ unknowns $x_2, x_3, \cdots, x_n$, and then applying the same elimination procedure to this system x_2 is eliminated from the last $n-2$ equations. So, we have equivalent system as follows,

$$
\begin{aligned}
a_{11}^{(1)}x_1 \quad + \quad a_{12}^{(1)}x_2 + \quad a_{13}^{(1)}x_3 \quad + \quad \cdots \quad + \quad a_{1n}^{(1)}x_n \quad &= \quad b_1^{(1)} \\
a_{22}^{(2)}x_2 + \quad a_{23}^{(2)}x_3 \quad + \quad \cdots \quad + \quad a_{2n}^{(2)}x_n \quad &= \quad b_2^{(2)}
\end{aligned}
$$

$$
\begin{aligned}
a_{33}^{(3)}x_3 \quad + \quad \cdots \quad + \quad a_{3n}^{(3)}x_n \quad &= \quad b_3^{(3)} \\
\cdots \qquad\qquad\qquad\qquad & \cdots \\
a_{n3}^{(3)}x_3 \quad + \quad \cdots \quad + \quad a_{nn}^{(3)}x_n \quad &= \quad b_n^{(3)}
\end{aligned}
\tag{5}
$$

where

$$
\begin{aligned}
m_{i2} &= -a_{i2}^{(2)}/a_{22}^{(2)} \\
a_{ij}^{(3)} &= a_{ij}^{(2)} + m_{i2}a_{2j}, \; b_i^{(3)} = b_i^{(2)} + m_{i2}b_2^{(2)}, \quad i,j = 3,4,\cdots,n.
\end{aligned}
\tag{6}
$$

Continuing this process repeatedly, we obtain the following equivalent system of equations at the $n-1$th step as

$$
\begin{aligned}
a_{11}^{(1)}x_1 + a_{12}^{(1)}x_2 + a_{13}^{(1)}x_3 + \cdots + a_{1n}^{(1)}x_n &= b_1^{(1)} \\
a_{22}^{(2)}x_2 + a_{23}^{(2)}x_3 + \cdots + a_{2n}^{(2)}x_n &= b_2^{(2)} \\
a_{33}^{(3)}x_3 + \cdots + a_{3n}^{(3)}x_n &= b_3^{(3)} \\
\cdots \qquad\qquad & \cdots \\
a_{nn}^{(n)}x_n &= b_n^{(n)}
\end{aligned}
\tag{7}
$$

$[U|C]$

where

$$\begin{aligned} m_{ik} &= -a_{ik}^{(k)}/a_{kk}^{(k)} \\ a_{ij}^{(k+1)} &= a_{ij}^{(k)} + m_{ik}a_{kj},\ b_i^{(k+1)} = b_i^{(k)} + m_{ik}b_k^{(k)}, \\ & \qquad i,j = k+1, k+2, \cdots, n;\ k = 1, 2, \cdots, n. \end{aligned} \tag{8}$$

The leading coefficients of the above system of equations (7) $a_{11}^{(1)}$, $a_{22}^{(2)}$, $\cdots$, $a_{nn}^{(n)}$, which are nonzero by assumption, are known as the *pivots* and the corresponding equations are called the *pivotal equations*.

We observe that the coefficient matrix of (7) is

$$U = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & \cdots & a_{2n}^{(2)} \\ 0 & 0 & a_{33}^{(3)} & \cdots & a_{3n}^{(3)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{nn}^{(n)} \end{bmatrix} \tag{9}$$

an upper triangular matrix. Hence, the value of its determinat is

$$\det U = a_{11}^{(1)} a_{22}^{(2)} a_{33}^{(3)} \cdots a_{nn}^{(n)} \tag{10}$$

which follows that $\det U \neq 0$ so that the system (7) has a unique solution.
Now the upper triangular system (7) can be easily solved as follows.

From the last equation we get $x_n = b_n^{(n)} / a_{nn}^{(n)}$; then we substitute this value of x_n in the last but one equation, we get the value of x_{n-1}, and then we substitute these values of x_n and x_{n-1} in the last but two equation and we compute x_{n-2}; and so on, finally we get x_1.

This process of solving an upper triangular system of linear equations is generally called *back substitution*.

Examples on Gauss's elimination method

i). Deduce the upper triangular matrix form the following matrix-

$$\begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & -2 \\ 3 & 2 & -4 \end{bmatrix}$$

ii). Solve the following system of equations

$$\begin{aligned} x &+ y - z = 2 \\ 2x &+ 3y + 5z = -3 \\ 3x &+ 2y - 3z = 6 \end{aligned}$$

by the Gauss elimination method.