Numerical Methods

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Outline of presentation

1. Newton's Method
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Newton's Method

Newton's method for finding a root of a differentiable function $f(x)$ is given by:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad (1)$$

$\Rightarrow f(x_n) = (x_n - x_{n+1}) f'(x_n)$

$f(x_n) = 0$

$x_n \to x_{n+1}$

We note that for the formula (1) to be well-defined, we must require that

$$f'(x_n) \neq 0, \quad \text{for any } x_n.$$

To provide us with a list of successive approximation, Newton's method (1) should be supplemented with one initial guess, say x_0.

The equation (1) will then provide the values of $x_1,\ x_2,\ x_3, \cdots$

One way of obtaining Newton's method is the following:

$x_0 = 1$

$x_{n+1} = x_n - \frac{x_n^3 - x_n - 4}{3x_n^2 - 1}$

Given a point x_n we are looking for the next point x_{n+1}.
A linear approximation of $f(x)$ at x_{n+1} is

$x_1 = 3$

$x_2 = \frac{29}{13} = 2.23$

$$f(x_{n+1}) \approx f(x_n) + (x_{n+1} - x_n)f'(x_n)$$

$x_3 =$

Since x_{n+1} should be an approximation to the root of $f(x)$, we set $f(x_{n+1}) = 0$, rearrange the terms and get (1).

Determine the roots correct to two decimal places using the Newton's method of the following equation

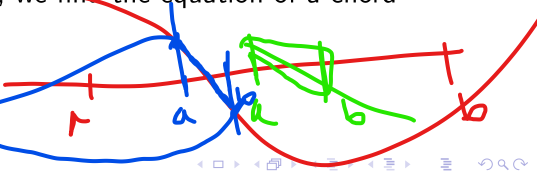
$$f(x) = x^3 - x - 4 = 0.$$

$f'(x) = 3x^2 - 1$

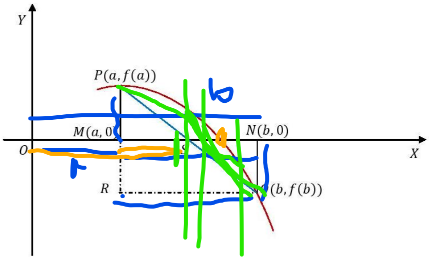
Regula Falsi method

Regula Falsi Method is the oldest method for finding the roots of a numerical equation $f(x) = 0$, also known as Method of False Position. In this method, we obtain two close points a and b such that $f(a)$ and $f(b)$ are of opposite signs. Eventually a root lies in between these points. Therefore the graph $y = f(x)$ must cross the $x-$axis at least once between a and b. As these two points are very close to each other, we may assume that the portion of the graph $y = f(x)$ between a and b is a straight line. The method of False position is entirely based on the validity of this particular assumption.

Now to derive the formula for evaluating the root, we find the equation of a chord joining the two points $(a, f(a))$ and $(b, f(b))$.



From the following figure , we have



$$
\begin{aligned}
\overline{OM} &= a, \quad \overline{ON} = b, \\
\overline{PM} &= |f(a)|, \quad \overline{NQ} = |f(b)|, \\
\overline{RQ} &= \overline{MN} = b - a, \\
\overline{MS} &= \overline{OS} - \overline{OM} = x - a.
\end{aligned}
$$

Now from the similar triangles $\triangle PMS$ and $\triangle PRQ$, we get

$$\frac{\overline{MS}}{\overline{PM}} = \frac{\overline{RQ}}{\overline{PR}} \implies \frac{x-a}{|f(a)|} = \frac{b-a}{|f(a)|+|f(b)|}$$

$$\therefore x = a + \frac{|f(a)|}{|f(a)|+|f(b)|}(b-a) \tag{2}$$

The value of x is not the true value of the root, because the graph $y = f(x)$ is not perfectly a straight line between a and b, rather it is a closer approximation to the true root. This is the first approximation of root and hence we denote this x by $x^{(1)}$.
If $f(x^{(1)})$ and $f(a)$ are of opposite signs, then the root lies between a and $x^{(1)}$ and we replace b by $x^{(1)}$ in (2) and obtain the next approximation $x^{(2)}$. Otherwise, we replace a by $x^{(1)}$ and generate the next approximation. This method is also called **linear interpolation method or chord method**.

Example: 1

Find a real root of the equation $f(x) = x^3 - 2x - 5 = 0$ by Regula Falsi method.

Solution: We see that $f(2) = -1$ and $f(3) = 16$, hence the root lies between 2 and 3. Let $a = 2$ and $b = 3$, then

$$x^{(1)} = a + \frac{|f(a)|}{|f(a)| + |f(b)|}(b - a) = 2 + \frac{1}{1 + 16}(3 - 2) = \frac{35}{17}$$

$$\therefore \; x^{(1)} = 2.058823529$$

Now, $f(x^{(1)}) = f(2.058823529) = -0.390799917 < 0$. Therefore, the root lies between 2.058823529 and 3. Again, using the formula, we get the second approximation as,

$$x^{(2)} = 2.058823529 + \frac{0.390799917}{0.390799917 + 16}(3 - 2.058823529)$$

$$= 2.08126366$$

Proceeding in this way, we get the next approximations as

$$
\begin{aligned}
x^{(3)} &= 2.089639211, \\
x^{(4)} &= 2.092739575, \\
x^{(5)} &= 2.093883710, \\
x^{(6)} &= 2.094305452, \\
x^{(7)} &= 2.094460846
\end{aligned}
$$

Secand method

One drawback of Newton's method is that it is necessary to evaluate $f'(x)$ at various points, which may not be practical for some choices of f. The secant method avoids this issue by using a finite difference to approximate the derivative. As a result, $f(x)$ is approximated by a secant line through two points on the graph of f, rather than a tangent line through one point on the graph.
Since a secant line is defined using two points on the graph of $f(x)$, as opposed to a tangent line that requires information at only one point on the graph, it is necessary to choose two initial iterates x_0 and x_1.

Then, as in Newton's method, the next iterate x_2 is then obtained by computing the x-value at which the secant line passing through the points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ has a y-coordinate of zero. This yields the equation

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_1 - x_0) + f(x_1) = 0$$

which has the solution

$$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)}$$

which can be rewritten as follows:

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)} \\ &= \frac{x_1(f(x_1) - f(x_0)) - f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)} \\ &= \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} \end{aligned} \tag{3}$$

Iteration Method

In the previous methods, we have identified the interval in which the root of $f(x) = 0$ lies, we discuss the methods which require one or more starting values of x, which need not necessarily enclose the root of $f(x) = 0$. The iteration method is one such method, which requires one starting value of x. We can use this method, if we can express $f(x) = 0$, as

$$x = \phi(x) \tag{4}$$

We can express $f(x) = 0$, in the above form in more than one way also.
For example, the equation $x^3 + x^2 - 1 = 0$ can be expressed in the following ways.

$$\begin{aligned} x &= (1+x)^{-\frac{1}{2}} \\ x &= (1-x^3)^{\frac{1}{2}} \\ x &= (1-x^2)^{\frac{1}{3}} \end{aligned}$$

and os on.

Let x_0 be an approximation to the desired root ξ, which we can find graphically or otherwise. Substituting x_0 in right hand side of (4), we get the first approximation as

$$x_1 = \phi(x_0) \tag{2}$$

The successive approximations are given by

$$\begin{aligned} x_2 &= \phi(x_1) \\ x_3 &= \phi(x_2) \\ \vdots & \quad \vdots \\ x_n &= \phi(x_{n-1}) \end{aligned} \tag{3}$$

Note: The sequence of approximations $x_0, x_1, x_2, \cdots, x_n$ given by (3) converges to the root ξ in an interval I, if $|\phi'(x)| < 1$ for all x in I.

Example: 2

Using the method of iteration find a positive root between 0 and 1 of the equation $xe^x - 1 = 0$

Solution: The given equation can be written as $x = e^{-x}$.

$$\therefore \qquad \phi(x) = e^{-x}$$

Here $|\phi'(x)| < 1$ for all $x < 1$.
So, we can use iterative method.
Let $x_0 = 1$, hence we obtain accordingly

$$
\begin{aligned}
x_1 &= e^{-1} = \frac{1}{e} = 0.3678794 \\
x_2 &= e^{-0.3678794} = 0.6922006 \\
x_3 &= e^{-0.6922006} = 0.5004735 \\
x_4 &= e^{-0.5004735} = 0.6062372 \\
x_5 &= e^{-0.6062372} = 0.5454735 \\
x_6 &= e^{-0.5454735} = 0.5796135 \\
x_7 &= e^{-0.5796135} = 0.5601496 \\
x_8 &= e^{-0.5601496} = 0.5712104 \\
x_8 &= e^{-0.5712104} = 0.5648712
\end{aligned}
$$

Proceeding like this, we get the required root as $x = 0.5671$.

Convergence

A sequence of iterates $\{x_n\}$ is said to converge with order $p \geq 1$ to a point x^* if

$$|x_{n+1} - x^*| \leq c|x_n - x^*|^p, \quad n \geq 0$$

for some constant $c > 0$.

Note: If $p = 1$, the sequence is said to converge linearly to x^*, if $p = 2$, the sequence is said to converge quadratically and so on.

Exercises

Given the following equations :

(i) $x^4 - x - 10 = 0$,

(ii) $x - e^{-x} = 0$

determine the initial approximations for finding the smallest positive root. Use the following methods to find the root correct to three decimal places with the following methods:

(a) Secant method, (b) Regula-Falsi method, (c) Iteration method