Numerical Methods

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September 14, 2021

Outline of presentation

Properties and relations of operators

$\Delta f = f(x+h) - f(x)$

(i) Forward difference of a constant function is zero i.e., if $f(x) = c$, then $\Delta f(x) = 0$.

(ii) If $f(x)$ is any function and k is a constant, then $\Delta[kf(x)] = k\Delta f(x)$.

(iii) If $f(x), g(x)$ be two function, then $\Delta[f(x) \pm g(x)] = \Delta f(x) \pm \Delta g(x)$. This holds for finitely many functions.

(iv) The forward difference follows the laws of indicies,

$$\Delta^m \cdot \Delta^n f(x) = \Delta^{m+n} f(x).$$

$$\begin{aligned}
\Delta^m \cdot \Delta^n f(x) &= (\Delta \cdot \Delta \cdot \Delta \cdots m\text{times}) \times \\
&\quad (\Delta \cdot \Delta \cdot \Delta \cdots n\text{times}) \times f(x) \\
&= (\Delta \cdot \Delta \cdot \Delta \cdots (m+n)\text{times}) \times f(x) \\
&= \Delta^{m+n} f(x).
\end{aligned}$$

(v) $\Delta[f(x)\cdot g(x)] = f(x+h)\cdot\Delta g(x) + g(x)\cdot\Delta f(x)$
$= g(x+h)\cdot\Delta f(x) + f(x)\cdot\Delta g(x)$

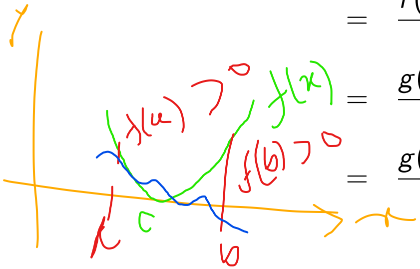
$$\begin{aligned}
\Delta[f(x)\cdot g(x)] &= f(x+h)g(x+h) - f(x)g(x) \\
&= f(x+h)g(x+h) + f(x+h)g(x) \\
&\quad - f(x+h)g(x) - f(x)g(x) \\
&= f(x+h)[g(x+h) - f(x)] + g(x)[f(x+h) - f(x)] \\
&= f(x+h)\cdot\Delta g(x) + g(x)\cdot\Delta f(x)
\end{aligned}$$

Again

$$\begin{aligned}
\Delta[f(x)\cdot g(x)] &= f(x+h)g(x+h) - f(x)g(x) \\
&= f(x+h)g(x+h) + g(x+h)f(x) \\
&\quad - g(x+h)f(x) - f(x)g(x) \\
&= g(x+h)[f(x+h) - f(x)] + f(x)[g(x+h) - g(x)] \\
&= g(x+h)\cdot\Delta f(x) + f(x)\cdot\Delta g(x).
\end{aligned}$$

(vi) $\Delta\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\cdot\Delta f(x) - f(x)\cdot\Delta g(x)}{g(x+h)g(x)}, \quad g(x) \neq 0.$

$$\begin{aligned}
\Delta\left[\frac{f(x)}{g(x)}\right] &= \frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)} = \frac{f(x+h)g(x) - g(x+h)f(x)}{g(x+h)g(x)} \\
&= \frac{f(x+h)g(x) + f(x)g(x) - f(x)g(x) - g(x+h)f(x)}{g(x+h)g(x)} \\
&= \frac{g(x)[f(x+h) - f(x)] - f(x)[g(x+h) - g(x)]}{g(x+h)g(x)} \\
&= \frac{g(x)\cdot\Delta f(x) - f(x)\cdot\Delta g(x)}{g(x+h)g(x)}
\end{aligned}$$



Transcendental and Polynomial equations

A polynomial equation of degree n will have exactly n roots, real or complex, simple or multiple. A transcendental equation may have one root or no root or infinite number of roots depending on the form of $f(x)$. The methods of finding the roots of $f(x) = 0$ are classified as,

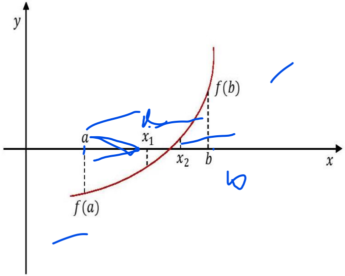
1. Direct Methods and
2. Numerical Methods.

There are no direct methods for solving higher degree algebraic equations or transcendental equations. If a and b are two numbers such that $f(a)$ and $f(b)$ have opposite signs, then a root of $f(x) = 0$ lies in between a and b. We take a or b or any valve in between a or b as first approximation x_1. This is further improved by numerical methods.

Here we discuss few important numerical methods to find a root of $f(x) = 0$.

Bisection Method

Identify two points $x = a$ and $x = b$ such that $f(a)$ and $f(b)$ are having opposite signs. Let $f(a)$ be negative and $f(b)$ be positive. Then there will be a root of $f(x) = 0$ in between a and b.



Let the first approximation be the mid point of the interval (a, b). i.e.

$$x_1 = \frac{a+b}{2}$$

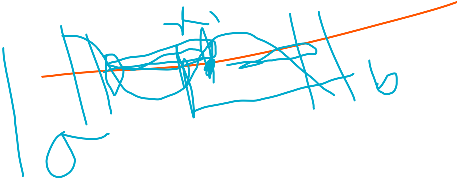
If $f(x_1) = 0$, then x_1 is a root, other wise root lies between a and x_1 or x_1 and b according as $f(x_1)$ is positive or negative.

Then again we bisect the interval and continue the process until the root is found to desired accuracy. Let $f(x_1)$ is positive, then root lies in between a and x_1.

The second approximation to the root is given by,

$$x_2 = \frac{a + x_1}{2}$$

If $f(x_2)$ is negative, then next approximation is given by



$$x_2 = \frac{x_1 + b}{2}$$

Similarly we can get other approximations. This method is also called Bolzano method.

Newton's Method

Newton's method for finding a root of a differentiable function $f(x)$ is given by:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1}$$

We note that for the formula (1) to be well-defined, we must require that

$$f'(x_n) \neq 0, \text{ for any } x_n.$$

To provide us with a list of successive approximation, Newton's method (1) should be supplemented with one initial guess, say x_0.

The equation (1) will then provide the values of $x_1,\ x_2,\ x_3, \cdots$

One way of obtaining Newton's method is the following:

Given a point x_n we are looking for the next point x_{n+1}.
A linear approximation of $f(x)$ at x_{n+1} is

$$f(x_{n+1}) \approx f(x_n) + (x_{n+1} - x_n)f'(x_n)$$

Since x_{n+1} should be an approximation to the root of $f(x)$, we set $f(x_{n+1}) = 0$, rearrange the terms and get (1).

Solve the following problems:

1. Find a root of $f(x) = xe^x - 1 = 0$, using Bisection method, correct to three decimal places.

2. Determine the roots correct to two decimal places using the Bisection method and Newton's method of the following equation

$$x^3 - x - 4 = 0.$$