Dynamical system, MSc, Semester III

Phonindra Nath Das

Department of Mathematics
Ramakrishna Mission Vivekananda Centenary College, Rahara

Email: *phonimath@gmail.com*

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Highlights

1. Flows, trajectories and orbits
2. Application

The time-evolutionary process may be described as a flow of a vector field. Generally, flow is frequently used for discussing the dynamics as a whole rather than the evolution of a system at a particular point. The solution $\underset{\sim}{x}(t)$ of a system $\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x})$ which satisfies $\underset{\sim}{x}(t_0) = \underset{\sim}{x}_0$ gives the past $(t < t_0)$ and future $(t > t_0)$ evolutions of the system. Mathematically, the flow is defined by $\phi_t(\underset{\sim}{x}) : U \to \mathbb{R}^n$ where $\phi_t(\underset{\sim}{x}) = \phi(t, \underset{\sim}{x})$ is a smooth vector function of $\underset{\sim}{x} \in U \subseteq \mathbb{R}^n$ and $t \in I \subseteq \mathbb{R}$ satisfying the equation

Flow

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_t(\underset{\sim}{x}) = \underset{\sim}{f}(\phi_t(\underset{\sim}{x}))$$

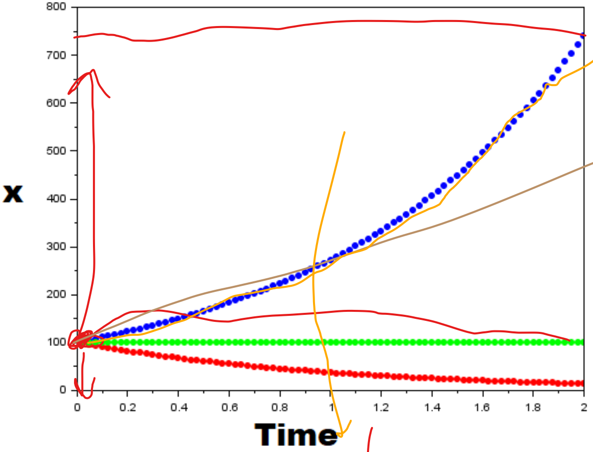
for all t such that the solution through $\underset{\sim}{x}$ exists and $\phi(0, \underset{\sim}{x}) = \underset{\sim}{x}$. The flow $\phi_t(\underset{\sim}{x})$ satisfies the following properties:

(a) $\phi_o = I_d$, (b) $\phi_{t+s} = \phi_t \circ \phi_s$.

Flows in $\mathbb{R}$: Consider a one-dimensional autonomous system represented by $\dot{x} = f(x), x \in \mathbb{R}$. We can imagine that a fluid is flowing along the real line with local velocity $f(x)$. This imaginary fluid is called the **phase fluid** and the real line is called the **phase line**. For solution of the system $\dot{x} = f(x)$ starting from an arbitrary initial position x_0, we place an imaginary particle, called a **phase point**, at x_0 and watch how it moves along with the flow in phase line in varying time t. As time goes on, the phase point (x, t) in the one-dimensional system $\dot{x} = f(x)$ with $x(0) = x_0$ moves along the x-axis according to some function $\phi(t, x_0)$. The function $\phi(t, x_0)$ is called the **trajectory** for a given initial state x_0, and the set $\{\phi(t, x_0) | t \in I \subseteq \mathbb{R}\}$ is the orbit of $x_0 \in \mathbb{R}$. The set of all qualitative trajectories of the system is called **phase portrait**.

Flows in $\mathbb{R}^2$: Consider a two-dimensional system represented by the following equations $\dot{x} = f(x, y)$, $\dot{y} = g(x, y)$, $(x, y) \in \mathbb{R}^2$. An imaginary fluid particle flows in the plane $\mathbb{R}^2$, known as phase plane of the system. The succession of states given parametrically by $x = x(t)$, $y = y(t)$ trace out a curve through some initial point $P(x(t_0), y(t_0))$ is called a **phase path**. The set $\{\phi(t, \underset{\sim}{x}_0)|t \in I \subseteq \mathbb{R}\}$ is the orbit of $\underset{\sim}{x}_0$ in $\mathbb{R}^2$. There are an infinite number of trajectories that would fill the phase plane when they are plotted. But the qualitative behavior can be determined by plotting a few trajectories with different initial conditions. The phase portrait displays how the qualitative behavior of a system is changing as x and y varies with time t. An orbit is called periodic if $x(t + p) = x(t)$ for some $p > 0$, for all t. The smallest integer p for which the relation is satisfied is called the prime period of the orbit. Flows in $\mathbb{R}$ cannot have oscillatory or closed path.

$r < 0$ $r = 0$ $r > 0$

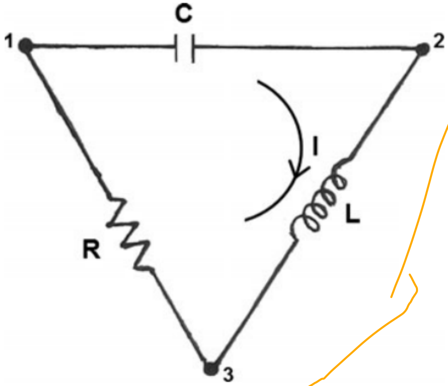


dx/dt = rx

x = ce^{rt}

Malthus model in population growth

First model!!



LCR circut model in physics

$$\frac{dV}{dt} = \frac{I}{C} \text{ and } \frac{dI}{dt} = -\frac{R}{L} I - \frac{V}{L}.$$

Guess what!

$$
\begin{aligned}
\frac{dx}{dt} &= ax - bxy \\
\frac{dy}{dt} &= -cy + exy
\end{aligned}
$$

where x, y are variables and a, b, c, e are constants.

Now we are in position to study the stability of such a system!!