Elastic waves:The action of a sudden disturbance in an elastic medium is transmitted at once to the other part of a body. At the beginning the remote parts of the body remains undisturbed and the deformation produced at a point are propagated through the body in the form of waves known as elastic waves.

Solution of one-dimensional wave equation

The one-dimensional wave equation may be given as

$$\frac{\partial^2\Phi}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}, \tag{1}$$

where x is the distance travelled by the wave and t is the time taken. $\Phi(u,v)$ is the disturbance and is the function of x and t. c is the velocity of wave.

To solve (1), D' Alembert's method may be taken into account, according to which it may be assumed that

$$u=x-ct,\ v=x+ct. \tag{2}$$

Using (2) in (1) gives,

$$\frac{\partial\Phi}{\partial x}=\frac{\partial\Phi}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial\Phi}{\partial v}\frac{\partial v}{\partial x}=\frac{\partial\Phi}{\partial u}+\frac{\partial\Phi}{\partial v}\qquad\left(\because\frac{\partial u}{\partial x}=1,\ \frac{\partial v}{\partial x}=1\right),$$

and

$$\frac{\partial^2\Phi}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial\Phi}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial\Phi}{\partial u}+\frac{\partial\Phi}{\partial v}\right)=\frac{\partial}{\partial u}\left(\frac{\partial\Phi}{\partial u}+\frac{\partial\Phi}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial\Phi}{\partial u}+\frac{\partial\Phi}{\partial v}\right)\frac{\partial v}{\partial x},$$

$$=\left(\frac{\partial^2\Phi}{\partial u^2}+\frac{\partial^2\Phi}{\partial u\partial v}\right)+\left(\frac{\partial^2\Phi}{\partial v\partial u}+\frac{\partial^2\Phi}{\partial v^2}\right).$$

Therefore,

$$\frac{\partial^2\Phi}{\partial x^2}=\frac{\partial^2\Phi}{\partial u^2}+2\frac{\partial^2\Phi}{\partial u\partial v}+\frac{\partial^2\Phi}{\partial v^2}. \tag{3}$$

Also,

$$\frac{\partial\Phi}{\partial t}=\frac{\partial\Phi}{\partial u}\frac{\partial u}{\partial t}+\frac{\partial\Phi}{\partial v}\frac{\partial v}{\partial t}=\frac{\partial\Phi}{\partial u}(-c)+\frac{\partial\Phi}{\partial v}(c)\qquad\left(\because\frac{\partial u}{\partial t}=-c,\ \frac{\partial v}{\partial t}=t\right).$$

This implies that,

$$\frac{\partial^2\Phi}{\partial t^2}=\frac{\partial}{\partial t}\left(\frac{\partial\Phi}{\partial t}\right)=\frac{\partial}{\partial u}\left(-c\frac{\partial\Phi}{\partial u}+c\frac{\partial\Phi}{\partial v}\right)\frac{\partial u}{\partial t}+\frac{\partial}{\partial v}\left(-c\frac{\partial\Phi}{\partial u}+c\frac{\partial\Phi}{\partial v}\right)\frac{\partial v}{\partial t}$$

$$=\left(-c\frac{\partial^2\Phi}{\partial u^2}+c\frac{\partial^2\Phi}{\partial u\partial v}\right)(-c)+\left(-c\frac{\partial^2\Phi}{\partial v\partial u}+c\frac{\partial^2\Phi}{\partial v^2}\right)(c),$$

$$\therefore \frac{\partial^2\Phi}{\partial t^2} = c^2\left(\frac{\partial^2\Phi}{\partial u^2} - 2\frac{\partial^2\Phi}{\partial u\partial v} + \frac{\partial^2\Phi}{\partial v^2}\right). \qquad (4)$$

Substituting (3) and (4) in (1), we get

$$\frac{\partial^2\Phi}{\partial u^2} + 2\frac{\partial^2\Phi}{\partial u\partial v} + \frac{\partial^2\Phi}{\partial v^2} = \frac{c^2}{c^2}\left(\frac{\partial^2\Phi}{\partial u^2} - 2\frac{\partial^2\Phi}{\partial u\partial v} + \frac{\partial^2\Phi}{\partial v^2}\right),$$

or, $\frac{\partial^2\Phi}{\partial u\partial v} = 0$. (5)

Integrating with respect to u, we may get

$\frac{\partial\Phi}{\partial v} = \chi(v) =$ (a function not containing u). (6)

Again, integrating with respect to v gives

$\Phi(u,v) = \int \chi(v) + f(u) = f(u) + g(v)$ (where, $f(u)$ is a function not containing v). (7)

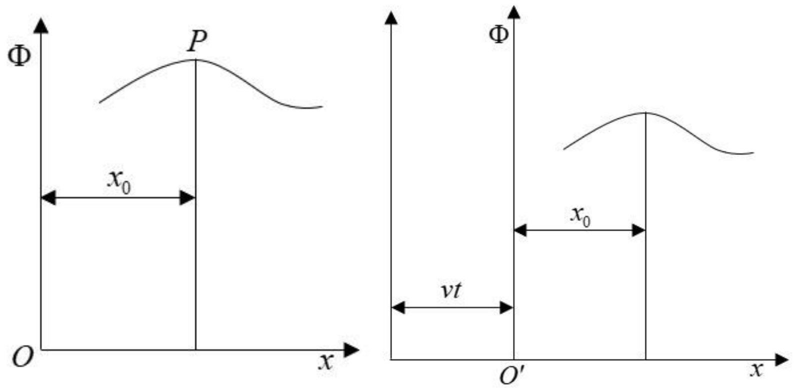
Therefore,

$$\Phi(x,t) = f(x-ct) + g(x+ct), \qquad (8)$$

where f and g are quite arbitrary.

Physical meaning of the solutions

$\Phi(x,t) = f(x-ct) \quad \Rightarrow$ at $t=0,\ \Phi = f(x)$ (9)



(a) (b)

Equation (9) is the equation of a curve on the (x,Φ) plane referred to a rectangular system of axes through a fixed point O. This curve is known as ***wave profile***.

Now, assume that the wave profile is moving with velocity v, so that at time t, the wave profile will move to a distance vt.

Let us again assume that the wave profile is moving along the positive direction of $x-$axis.

Let us assume that the amplitude of the wave profile in the two diagrams are same throughout the motion, i.e. the position of the wave profile at $t=0$ and $t=t_1$ remains the same.

$$\left[f\left(x-ct\right)\right]_{x=x_0,t=0}=\left[f\left(x-ct\right)\right]_{x=x_0+vt_1,t=t_1},$$

or, $f\left(x_0\right)=f\left(x_0+vt_1-ct_1\right)$,

or, $x_0=x_0+vt_1-ct_1$,

or, $v=c$, (10)

which means that the wave profile velocity $(\equiv v)$ is equal to the velocity of the wave $(\equiv c)$.

The decrease of amplitude of a wave as it travels is called ***attenuation*** and the change of shape is called ***distortion***.

The physical meaning of $\Phi=f\left(x-ct\right)$ is that it is a travelling wave (or a progressive wave) which travels in the positive direction of $x-$axis with a constant speed c, without attenuation or distortion.

The physical meaning of $\Phi=g\left(x+ct\right)$ is that it is a travelling wave (or a progressive wave) which travels in the negative direction of $x-$axis with a constant speed c, without attenuation or distortion.

Plane wave: When a propagating disturbance is confined to a plane throughout the motion, the associated wave is known as plane wave.

Waves in three dimensions

The wave equation in three dimensions may be written as:

$$\nabla^2\Phi=\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}, \tag{1}$$

where $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$, c is the wave velocity in meters/sec and t is the time in sec.

(x,y,z) are coordinates in meters/sec. Φ is the disturbance and the function of x,y,z and t.

To solve the equation (1), we consider

$$\Phi=f\left(lx+my+nz+st\right), \tag{2}$$

where (l,m,n) are the direction cosines of the normal to the plane at a distance X from the origin O and is an arbitrary function.

Using (2) in (1) yields

$$l^2f''+m^2f''+n^2f''=\frac{1}{c^2}s^2f'',$$

or, $\left(l^2+m^2+n^2\right)f''=\frac{1}{c^2}s^2f''$,

or, $f''=\frac{1}{c^2}s^2f''$, $\left(\because l^2+m^2+n^2=1\right)$

or, $c^2=s^2$.

Hence, the general solution is

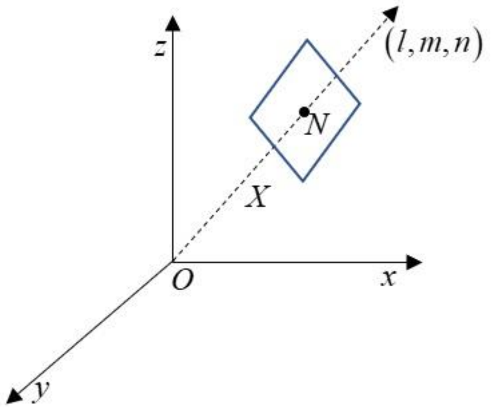
$\Phi(x,y,z,t)$
$=f\left(lx+my+nz-ct\right)+g\left(lx+my+nz+ct\right)$, (3)

where f and g are arbitrary functions.

Let $X=lx+my+nz$, (4)

then from (3), we have

$\Phi=f\left(X-ct\right)+g\left(X+ct\right)$. (5)



$f\left(X-ct\right)\equiv f\left(lx+my+nz-ct\right)$ is a progressive wave in three dimensions travelling along the normal to the plane (i.e. ON), $lx+my+nz=X$ with uniform velocity c.

Similarly, $g\left(X+ct\right)\equiv f\left(lx+my+nz+ct\right)$ is a progressive wave in three dimensions travelling along the normal to the plane (i.e. NO), $lx+my+nz=X$ with uniform velocity c.

Propagation of waves in elastic media

1. Waves of dilatation and waves of distortion in isotropic media
   a) The equation of motion without body force is

$$\left(\lambda+\mu\right)\frac{\partial\Delta}{\partial x}+\mu\nabla^2u=\rho\frac{\partial^2u}{\partial t^2}, \quad (6)$$

$$\left(\lambda+\mu\right)\frac{\partial\Delta}{\partial y}+\mu\nabla^2v=\rho\frac{\partial^2v}{\partial t^2}, \quad (7)$$

$$\left(\lambda+\mu\right)\frac{\partial\Delta}{\partial z}+\mu\nabla^2w=\rho\frac{\partial^2w}{\partial t^2}, \quad (8)$$

where $\left(u,v,w\right)$ are component of displacement and $\Delta=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}$ is the dilatation.

$\left(\lambda,\mu\right)$ are lame constant and ρ is material density.

Differentiating (6), (7) and (8) with respect to x,y and z respectively and adding, gives

$$\left\{(\lambda+\mu)\frac{\partial^2\Delta}{\partial x^2}+\mu\nabla^2\left(\frac{\partial u}{\partial x}\right)\right\}+\left\{(\lambda+\mu)\frac{\partial^2\Delta}{\partial y^2}+\mu\nabla^2\left(\frac{\partial v}{\partial y}\right)\right\}+\left\{(\lambda+\mu)\frac{\partial^2\Delta}{\partial z^2}+\mu\nabla^2\left(\frac{\partial w}{\partial z}\right)\right\},$$

$$=\rho\frac{\partial^2}{\partial t^2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\right)$$

or, $(\lambda+\mu)\left(\frac{\partial^2\Delta}{\partial x^2}+\frac{\partial^2\Delta}{\partial y^2}+\frac{\partial^2\Delta}{\partial z^2}\right)+\mu\nabla^2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=\rho\frac{\partial^2}{\partial t^2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\right),$

or, $(\lambda+2\mu)\nabla^2\Delta=\rho\frac{\partial^2}{\partial t^2}\Delta\,,$

or, $\nabla^2\Delta=\frac{1}{c_1^{\,2}}\frac{\partial^2\Delta}{\partial t^2},\quad$ where $c_1^{\,2}=\frac{\lambda+2\mu}{\rho}$. (9)

Equations (2) and (3) shows that volume dilatation or compression is transmitted in the form of wave through the medium with a velocity $c_1=\sqrt{\frac{\lambda+2\mu}{\rho}}$. Such a wave is named as ***wave of dilatation***. This wave is also called ***P-wave or Primary wave***.

b) Differentiate (7) with respect to x and differentiate (6) with respect to y and then subtracting yields

$$\left\{(\lambda+\mu)\frac{\partial^2\Delta}{\partial x\partial y}+\mu\nabla^2\left(\frac{\partial v}{\partial x}\right)\right\}-\left\{(\lambda+\mu)\frac{\partial^2\Delta}{\partial y\partial x}+\mu\nabla^2\left(\frac{\partial u}{\partial y}\right)\right\}=\rho\frac{\partial^2}{\partial t^2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right),$$

or, $\mu\nabla^2\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)=\rho\frac{\partial^2}{\partial t^2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right),$

or, $\mu\nabla^2\tilde{\omega}_z=\rho\frac{\partial^2\tilde{\omega}_z}{\partial t^2},$

or, $\nabla^2\tilde{\omega}_z=\frac{1}{c_2^{\,2}}\frac{\partial^2\tilde{\omega}_z}{\partial t^2},$ (10)

where $c_2=\sqrt{\frac{\mu}{\rho}}$. (11)

Similarly, $\nabla^2\tilde{\omega}_x=\frac{1}{c_2^{\,2}}\frac{\partial^2\tilde{\omega}_x}{\partial t^2}$ and $\nabla^2\tilde{\omega}_y=\frac{1}{c_2^{\,2}}\frac{\partial^2\tilde{\omega}_y}{\partial t^2}$. (12)

where $\tilde{\omega}_x=\frac{1}{2}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right),\ \tilde{\omega}_y=\frac{1}{2}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial y}\right),\tilde{\omega}_z=\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$. (13)

Equation (10) and (12) show rotation disturbances are transmitted in the form of waves through the medium with a velocity $c_2 = \sqrt{\mu/\rho}$. Such a wave is named as ***wave of distortion***. This wave is also called as ***S-wave or secondary wave***.

c) If $\Delta = 0$, then from (5), we have

$$\mu\nabla^2 u = \rho\frac{\partial^2 u}{\partial t^2}, \tag{14}$$

or, $\nabla^2 u = \frac{1}{c_2^{\,2}}\frac{\partial^2 u}{\partial t^2}$, where $c_2 = \sqrt{\mu/\rho}$.

Similarly, the other equations

$$\nabla^2 v = \frac{1}{c_2^{\,2}}\frac{\partial^2 v}{\partial t^2} \text{ and } \nabla^2 w = \frac{1}{c_2^{\,2}}\frac{\partial^2 w}{\partial t^2}. \tag{15}$$

$\Delta = 0 \Rightarrow$ there is neither volume expansion or compression, hence, it is called equivoluminal waves. This wave is also called the ***distortion wave*** or ***Secondary wave*** or ***S-wave***.

If the rotation components are zero, then we have

$$\tilde{\omega}_x = \tilde{\omega}_y = \tilde{\omega}_z = 0,$$

$$\tilde{\omega}_x = 0 \Rightarrow \frac{\partial w}{\partial y} = \frac{\partial v}{\partial z}. \tag{16}$$

Similarly, we have $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$ and $\frac{\partial w}{\partial x} = \frac{\partial u}{\partial z}$. (17)

Considering, $u = \frac{\partial \Phi}{\partial x}$, $v = \frac{\partial \Phi}{\partial y}$ and $w = \frac{\partial \Phi}{\partial z}$, we see that the equation (16) and (17) is satisfied.

Now, $\Delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = \nabla^2 \Phi$. (18)

Considering (16), (17) and (18), the equations of motion become

$$(\lambda + \mu)\frac{\partial}{\partial x}\left(\nabla^2 \Phi\right) + \mu\nabla^2\left(\frac{\partial \Phi}{\partial x}\right) = \rho\frac{\partial^2}{\partial t^2}\left(\frac{\partial \Phi}{\partial x}\right),$$

$$\frac{\partial}{\partial x}\left((\lambda + \mu)\nabla^2 \Phi + \mu\nabla^2 \Phi - \rho\frac{\partial^2 \Phi}{\partial t^2}\right) = 0,$$

$$(\lambda + 2\mu)\nabla^2 \Phi = \rho\frac{\partial^2 \Phi}{\partial t^2},$$

$$\nabla^2\Phi = \frac{1}{c_1^{\,2}}\frac{\partial^2\Phi}{\partial t^2}, \tag{19}$$

where $c_1 = \sqrt{\dfrac{\lambda+2\mu}{\rho}}$. (20)

In this case, the motion is irrotational. Hence, the wave is known as irrotational waves. It is also called as ***dilatational wave*** or ***Primary wave*** or ***P-wave***.

Body waves

Here, we consider the propagation of wave in an unbounded medium. Consider the displacement components as

$$u = F_1(lx+my+nz-ct),\quad v = F_2(lx+my+nz-ct),\quad w = F_3(lx+my+nz-ct), \tag{1}$$

where (l,m,n) are the direction cosines of the line of wave propagation and c is the velocity of wave. The equation of motion without body forces are

$$(\lambda+\mu)\frac{\partial\Delta}{\partial x} + \mu\nabla^2 u = \rho\frac{\partial^2 u}{\partial t^2}, \tag{2}$$

$$(\lambda+\mu)\frac{\partial\Delta}{\partial y} + \mu\nabla^2 v = \rho\frac{\partial^2 v}{\partial t^2}, \tag{3}$$

$$(\lambda+\mu)\frac{\partial\Delta}{\partial z} + \mu\nabla^2 w = \rho\frac{\partial^2 w}{\partial t^2}, \tag{4}$$

From (1) and (2), we have

$$\Delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = lF_1' + mF_2' + nF_3', \tag{5}$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = (l^2+m^2+n^2)F_1'' = F_1'' \quad (\because l^2+m^2+n^2=1). \tag{6}$$

$$\frac{\partial\Delta}{\partial x} = l^2F_1'' + lmF_2'' + lnF_3'',$$

Therefore, equations (2), (3) and (4) yield

$$(\lambda+\mu)\left(l^2F_1'' + lmF_2'' + lnF_3''\right) + \mu F_1'' = \rho c^2 F_1'',$$

$$\left((\lambda+\mu)l^2 + \mu - \rho c^2\right)F_1'' + lm(\lambda+\mu)F_2'' + ln(\lambda+\mu)F_3'' = 0. \tag{7}$$

Similarly,

$$lm(\lambda+\mu)F_1'' + \left((\lambda+\mu)m^2 + \mu - \rho c^2\right)F_2'' + mn(\lambda+\mu)F_3'' = 0, \tag{8}$$

$$ln(\lambda+\mu)F_1''+mn(\lambda+\mu)F_2''+\left((\lambda+\mu)n^2+\mu-\rho c^2\right)F_3''=0. \tag{9}$$

Now, eliminating F_1'', F_2'' and F_3'' from (8), (9) and (10), we have

$$\begin{vmatrix} (\lambda+\mu)l^2+\mu-\rho c^2 & lm(\lambda+\mu) & ln(\lambda+\mu) \\ lm(\lambda+\mu) & (\lambda+\mu)m^2+\mu-\rho c^2 & mn(\lambda+\mu) \\ ln(\lambda+\mu) & mn(\lambda+\mu) & (\lambda+\mu)n^2+\mu-\rho c^2 \end{vmatrix}=0.$$

This is a cubic equation in c^2.

$$\Rightarrow\left(\mu-\rho c^2\right)^2\left(\lambda+2\mu-\rho c^2\right)=0,$$

$$\therefore c^2=\frac{\mu}{\rho} \text{ or } c^2=\frac{\lambda+2\mu}{\rho}.$$

P-waves are called body waves in an unbounded medium. If we consider any boundary/ interface then another type of wave came into existence, which is named as surface waves. The existence of surface waves is near to the surface of the elastic body. The surface waves vanishes if we move interior of a body.

Elastic waves and their properties

A *wave* is a disturbance in a medium that travels outward from its source. It travels from one place to another by means of a medium, but the medium itself is not transported. All material media viz. solids, liquids, and gases, can carry energy and information by means of waves. *Waves* may be classified according to the physical properties of the medium and in many other ways as direction of particle motion, periodicity, shape of wave fronts, and number of dimensions. *Elastic waves* are mechanical waves propagating in an elastic medium as an effect of forces associated with volume deformation and shape deformation of medium elements. In an absolutely precise way, *elastic waves* are waves propagated by the elastic deformation of a medium. *Seismic waves* are waves of energy that travel through the Earth. It is as a result of an earthquake, explosion, or some other process that imparts low-frequency acoustic energy which propagate through the Earth's interior or along its surface layers. In other words, *seismic waves* are elastic waves generated by an impulse such as an earthquake or an artificial explosion. Seismic waves are primarily studied by seismologists and geophysicists. Seismic wavefields are measured by a seismograph, geophone, hydrophone (in water), or accelerometer. *Seismology* is the scientific discipline that primarily deals with the study of earthquakes and propagation of seismic waves through the Earth. Principal types of elastic waves can be displayed by the flow chart as sketched out in Fig. 1.1.

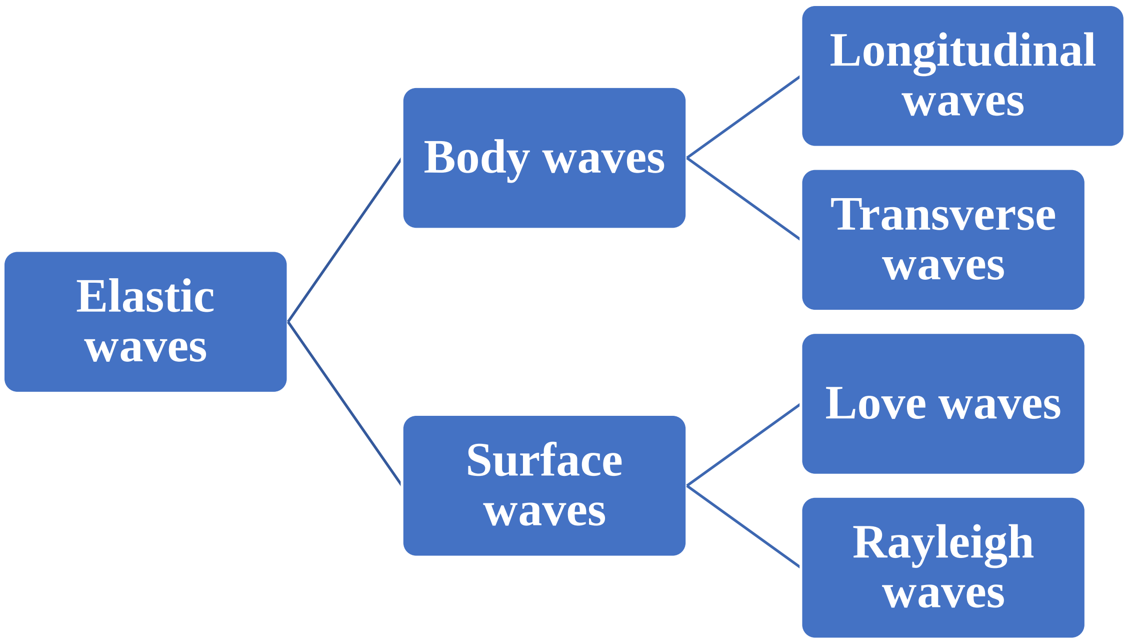


Fig. 1.1 Hierarchical overview of elastic waves.

Body waves: *Body waves* propagate through the interior of an elastic medium. Body waves may be considered as free waves as they have freedom to propagate in practically every direction through the medium. Further, these consist of two types of waves viz. *longitudinal waves* and *transverse waves*.

Longitudinal waves: *Longitudinal waves* are also known as *compressional* or *irrotational* or *pressure* or *dilatational waves*. These waves are also called *P-waves* (primary waves) in seismology as they represent the first waves appearing on seismograms. In longitudinal wave, the direction of particle motion is same or opposite to the direction of wave propagation. The particle motion consists of alternating compression and dilatation so that the material returns to its original shape after wave passes. These waves can propagate through any type of material i.e. solid, liquid or gaseous as they are longitudinal.

Transverse waves: *Transverse waves* are also known as *shear* or *rotational* or *equivoluminal* or *distortional waves*. These waves are also called *S-waves* (secondary waves) in seismology as they arrive second in a seismic station. In transverse wave, the direction of particle motion is perpendicular to the direction of wave propagation. These waves can propagate only through solids as fluids (liquids and gases) do not support shear stress. If during the passage of a transverse wave the particle motion is confined to a particular plane only, then it is said to be polarised transverse wave. Moreover, S-waves are classified as SH-waves and SV-waves. A horizontally travelling wave so polarised that the particle motion is only in the horizontal plane is called *SH-wave*. If the polarization is in the vertical plane, then the wave is called *SV-wave*. The particle motion of P-wave and S-wave may be illustrated by Fig. 1.2.

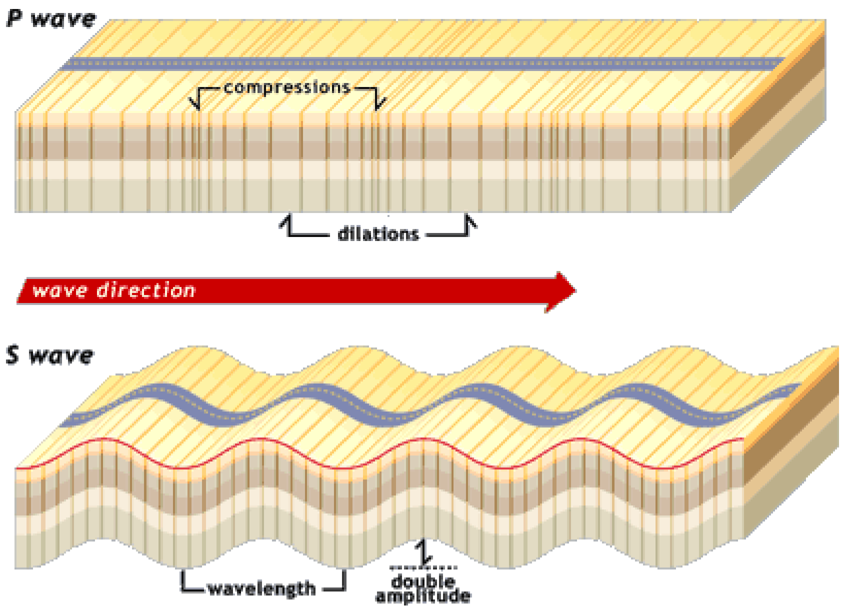


Fig. 1.2 Particle motion of P-wave and S-wave.

Surface waves: *Surface waves* propagate along the surface of an elastic medium. Owing to their low frequency, long duration and large amplitude, they can be the most destructive type of elastic wave. Surface waves can be considered as bound waves as they are bound to some surface or some layer during their propagation. If there are boundaries or interfaces, then surface wave will propagate. The effect of surface waves decay as the depth increases. Further, surface waves comprise of two types of waves viz. *Love waves* and *Rayleigh waves*.

Love waves: *Love waves* are horizontally polarized surface waves possessing the fact that particle motion takes place only in the horizontal plane and is transverse to the direction of propagation of wave. One of the phenomenal characteristics of Love waves are that these waves are noticed only when shear wave velocity in the substrate is more than that in the superficial layer.These waves are named after Augustus Edward Hough Love, a British mathematician who developed a mathematical model of the wave in 1911. Love waves have the largest amplitude. Love waves are dispersive in nature i.e. the wave velocity is dependent on frequency.

Rayleigh waves: *Rayleigh waves* are elastic surface waves that are elliptically polarised in the plane which is determined by the normal to the surface and by the direction of wave propagation. These waves propagate along the stress free boundary of an elastic half-space such that the disturbance is largely confined to the neighbourhood of the free boundary of the half-space. Rayleigh waves are combination of both longitudinal and transverse waves. As these waves are both longitudinal and transverse, the surface is moving in a vertical circular path-back and forth, and up and down. Hence, near the surface of a homogeneous elastic half-space, the particle motion is a retrograde vertical ellipse i.e. anticlockwise for a wave travelling to the right. Rayleigh waves are analogous to ocean waves. The existence of these waves was predicted by Lord Rayleigh in 1885and, therefore, was named as Rayleigh waves. Rayleigh waves are also dispersive in nature. The speed of propagation of P-wave, S-wave, Love wave and Rayleigh wave may be represented as $c_P > c_S > c_L > c_R$. Immense information pertaining to elastic waves can be extracted from distinct available monographs (Love, 1944; Ewing *et al.*, 1957; Achenbach, 1976; Udias, 1999). The particle motion of Love wave and Rayleigh wave may be depicted by Fig. 1.3.

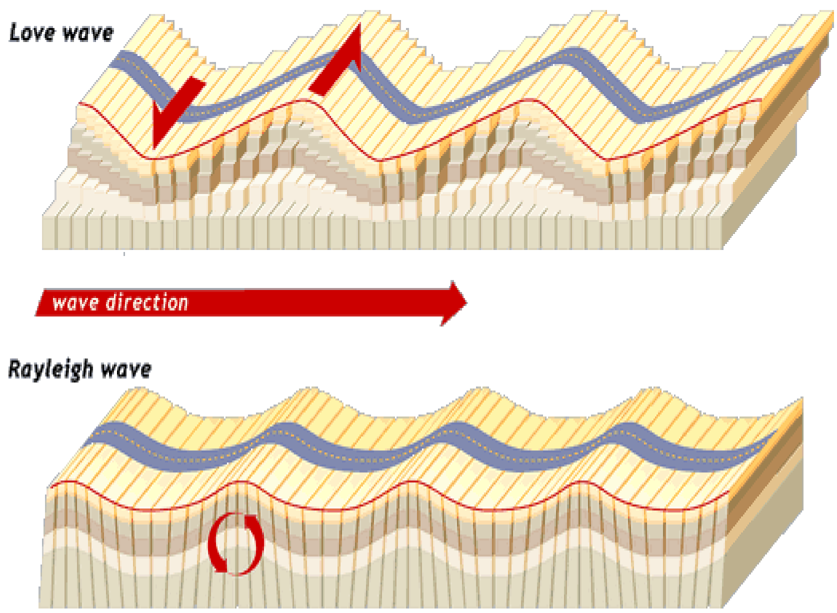


Fig. 1.3 Particle motion of Love wave and Rayleigh wave.

Polarized transverse waves: If during the passage of transverse wave the particle motion is confined to a particular plane only, then the wave is said to be polarized transverse wave.

A horizontally travelling wave so polarized that the particle motion is only in a horizontal plane, then it is called SH-wave.

If the polarization is in vertical plane, it is called SV-wave.

Propagation of Love in a superficial layer lying over a semi-infinite half-space

Love wave: A.E.H Love in 1911. It is a horizontally polarized wave. Also known as Shear wave or SH-wave.

For the propagation of Love waves, a superficial layer is essential. We consider the thickness of the layer as H.

Let the (rigidity, density, displacement) of the upper and lower medium be (μ_1, ρ_1, v_1) and (μ_2, ρ_2, v_2), respectively.

Let us assume that $x-$axis is along the direction of wave propagation and $z-$axis is positive pointing downwards. For the propagation of Love wave, we have

$$u = w = 0,\ v = v(x,z,t). \tag{1}$$

The displacement is independent of y.

At first we look for the equation governing the propagation of Love wave in homogeneous isotropic elastic medium. Since the equations of motion for a homogeneous isotropic elastic solid in the absence of body forces are

$$\tau_{ij} = \rho \ddot{u}_i \qquad \text{(i, j=1,2,3)}$$

In component form, above equation can be written as

$$\frac{\partial}{\partial x}\tau_{xx} + \frac{\partial}{\partial y}\tau_{xy} + \frac{\partial}{\partial z}\tau_{xz} = \rho\frac{\partial^2 u}{\partial t^2}, \tag{2}$$

$$\frac{\partial}{\partial x}\tau_{xy} + \frac{\partial}{\partial y}\tau_{yy} + \frac{\partial}{\partial z}\tau_{yz} = \rho\frac{\partial^2 v}{\partial t^2}, \tag{3}$$

$$\frac{\partial}{\partial x}\tau_{xz} + \frac{\partial}{\partial y}\tau_{yz} + \frac{\partial}{\partial z}\tau_{zz} = \rho\frac{\partial^2 w}{\partial t^2}. \tag{4}$$

Hooke's law in isotropic medium gives us the equation:

$\tau_{ij} = \lambda \Delta \delta_{ij} + 2\mu \varepsilon_{ij}$,

where λ, μ are Lame's constant and Δ is cubical dilatation.

Since,

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).$$

Calculating the strain components:

$$\varepsilon_{xx} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x} \right) = \frac{\partial u}{\partial x} = 0,\ \varepsilon_{yy} = \frac{1}{2}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial y} \right) = \frac{\partial v}{\partial y} = 0,\ \varepsilon_{zz} = \frac{1}{2}\left(\frac{\partial w}{\partial z} + \frac{\partial w}{\partial z} \right) = \frac{\partial w}{\partial z} = 0,$$

$$\varepsilon_{xy} = \varepsilon_{yx} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \frac{1}{2}\frac{\partial v}{\partial x},\ \varepsilon_{xz} = \varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) = 0,\ \varepsilon_{yz} = \varepsilon_{zy} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) = \frac{1}{2}\frac{\partial v}{\partial z}.$$

Therefore, the stress components may be calculated as follows:

$$\tau_{xx} = \lambda\Delta\delta_{xx} + 2\mu\varepsilon_{xx} = 0,\ \tau_{yy} = \lambda\Delta\delta_{yy} + 2\mu\varepsilon_{yy} = 0,\ \tau_{zz} = \lambda\Delta\delta_{zz} + 2\mu\varepsilon_{zz} = 0,$$

$$\tau_{xy} = \lambda\Delta\delta_{xy} + 2\mu\varepsilon_{xy} = \mu\frac{\partial v}{\partial x},\ \tau_{xz} = \lambda\Delta\delta_{xz} + 2\mu\varepsilon_{xz} = 0,\ \tau_{yz} = \lambda\Delta\delta_{yz} + 2\mu\varepsilon_{yz} = \mu\frac{\partial v}{\partial z}.$$

The equations of motion without body forces are:

$$(\lambda + \mu)\frac{\partial \Delta}{\partial x} + \mu\nabla^2 u = \rho\frac{\partial^2 u}{\partial t^2}, \quad (2)$$

$$(\lambda + \mu)\frac{\partial \Delta}{\partial y} + \mu\nabla^2 v = \rho\frac{\partial^2 v}{\partial t^2}, \quad (3)$$

$$(\lambda + \mu)\frac{\partial \Delta}{\partial z} + \mu\nabla^2 w = \rho\frac{\partial^2 w}{\partial t^2}. \quad (4)$$

where $\Delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$, as v is independent of y.

Therefore, equations (2) and (4) are identically satisfied. The only non-vanishing equation of motion is

$$\mu\nabla^2 v = \rho\frac{\partial^2 v}{\partial t^2},$$

or, $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2} = \frac{1}{\beta^2}\frac{\partial^2 v}{\partial t^2}$, where $\beta^2 = \frac{\mu}{\rho}$ and $\frac{\partial^2 v}{\partial y^2}$ vanishes as v is independent of y.

Now, for the upper layer, the non-vanishing equation of motion may be written as

$$\frac{\partial^2 v_1}{\partial x^2}+\frac{\partial^2 v_1}{\partial z^2}=\frac{1}{\beta_1^2}\frac{\partial^2 v_1}{\partial t^2}, \tag{5}$$

where $\beta_1{}^2=\frac{\mu_1}{\rho_1}$. (6)

To solve (5), we consider

$$v_1(x,z,t)=\vartheta_1(z)e^{ik(x-ct)}. \tag{7}$$

Using (7) in (5) yields

$$-k^2\vartheta_1 e^{ik(x-ct)}+\frac{d^2\vartheta_1}{dz^2}e^{ik(x-ct)}=-\frac{k^2c^2}{\beta_1^2}\vartheta_1 e^{ik(x-ct)},$$

$$\frac{d^2\vartheta_1}{dz^2}+\left(\frac{c^2}{\beta_1^2}-1\right)k^2\vartheta_1=0,$$

$$\frac{d^2\vartheta_1}{dz^2}+s_1{}^2\vartheta_1=0, \tag{8}$$

where $s_1=k\sqrt{\frac{c^2}{\beta_1^2}-1}$. (9)

The solution of equation (8) may be written as

$$\vartheta_1=A\cos s_1 z+B\sin s_1 z, \tag{10}$$

where A and B are arbitrary constants.

Therefore, from (7), we have

$$v_1=\left(A\cos s_1 z+B\sin s_1 z\right)e^{ik(x-ct)}. \tag{11}$$

The equation of motion for the lower medium is

$$\frac{\partial^2 v_2}{\partial x^2}+\frac{\partial^2 v_2}{\partial z^2}=\frac{1}{\beta_2{}^2}\frac{\partial^2 v_2}{\partial t^2}, \tag{12}$$

where $\beta_2{}^2=\frac{\mu_2}{\rho_2}$. (13)

To solve (12), we consider

$$v_2(x,z,t)=\vartheta_2(z)e^{ik(x-ct)}. \tag{14}$$

Using (14) in (12) yields

$$\frac{d^2\vartheta_2}{dz^2} = s_2{}^2\vartheta_2, \qquad (15)$$

where $s_2 = k\sqrt{1-\frac{c^2}{\beta_2{}^2}}$. (16)

The solution of equation (15) may be written as

$$\vartheta_2 = Ce^{-s_2 z} + De^{s_2 z}, \qquad (17)$$

where C and D are arbitrary constants.

Therefore, from (14), we have

$v_2 = Ce^{-s_2 z}e^{ik(x-ct)}$ (since, we are concerned with surface waves). (18)

Boundary conditions:

i. The upper surface is stress free
$(\tau_{yz})_1 = 0,$ at $z = -H$.

ii. The displacements are continuous at the interface
$v_1 = v_2,$ at $z = 0$.

iii. The stresses are continuous at the interface
$(\tau_{yz})_1 = (\tau_{yz})_2$, at $z = 0$.

Here, $(\tau_{yz})_1$ and $(\tau_{yz})_2$ are the non-vanishing shear stresses associated with the upper and lower medium, respectively.

The boundary condition (i)give

$$(\tau_{yz})_1 = \mu_1 \frac{\partial v_1}{\partial z} = 0, \quad \text{at } z = -H,$$

$$\Rightarrow \mu_1 s_1 (-A\sin s_1 z + B\cos s_1 z)e^{ik(x-ct)} = 0, \quad \text{at } z = -H,$$

$$\Rightarrow \mu_1 s_1 (A\sin s_1 H + B\cos s_1 H)e^{ik(x-ct)} = 0,$$

$$\Rightarrow A\sin s_1 H + B\cos s_1 H = 0,$$

$$\Rightarrow \tan s_1 H = -\frac{B}{A}. \qquad (19)$$

The boundary condition (ii) give

$$A = C. \qquad (20)$$

The boundary condition (iii) give

$$\mu_1 s_1 (-A\sin s_1 z + B\cos s_1 z)e^{ik(x-ct)} = -C\mu_2 s_2 e^{-s_2 z}e^{ik(x-ct)}, \quad \text{at } z = 0,$$

$$\Rightarrow \mu_1 s_1 B = -C\mu_2 s_2 ,$$

$$\Rightarrow \frac{B}{C} = -\frac{\mu_2 s_2}{\mu_1 s_1} , \tag{21}$$

Solving equations (19), (20) and (21) yield

$$\tan\left(kH\sqrt{\frac{c^2}{\beta_1^{\,2}} - 1} \right) = \frac{\mu_2\sqrt{1-\dfrac{c^2}{\beta_2^{\,2}}}}{\mu_1\sqrt{\dfrac{c^2}{\beta_1^{\,2}} - 1}} . \tag{22}$$

Equation (22) is the dispersion equation for the propagation of Love wave in a superficial layer lying over a semi-infinite half-space.

Validation:

$$\sqrt{\frac{c^2}{\beta_1^{\,2}} - 1} > 0 \quad \Rightarrow \frac{c^2}{\beta_1^{\,2}} > 1 \qquad \Rightarrow c > \beta_1 .$$

Similarly,

$$\sqrt{1-\frac{c^2}{\beta_2^{\,2}}} > 0 \quad \Rightarrow \frac{c^2}{\beta_2^{\,2}} < 1 \qquad \Rightarrow c < \beta_2 .$$

Therefore, it may be concluded that

$$\beta_1 < c < \beta_2 .$$